Packet Delay under the Golden Ratio Weighted TDM Policy in a Multiple-Access Channel

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Abstract—Consider n transmission stations sharing a single communication channel. Packets arrive at the stations according to n independent renewal processes, possibly with different rates. The transmitters are assumed to be able to store an unlimited number of packets in their buffers. The stations transmit packets during time slots allocated to them according to a given *conflict-free distributed protocol*. The cost criterion according to which protocols are evaluated is the long-run weighted average buffer occupancies. (The average waiting time is a special case of such a weighting.) A lower bound to the cost criterion under time division multiplexing (TDM) protocols is given, and the costs of two protocols are analyzed. The first protocol is the *random-control* policy, and the second is the *golden ratio* policy which is shown to achieve a cost close to the lower bound for realistic parameters.

I. Introduction

CONSIDER n transmission stations sharing a single communication channel. The time axis is divided into equal segments called slots. Messages are correspondingly "packetized"; the transmission time of a packet is one slot, and the transmission of a packet only starts on a slot boundary. Each station has an independent arrival process of packets and is capable of storing an unlimited number of them in its buffer. A transmission policy determines when a station with a nonempty packet queue may transmit.

In this paper we discuss only *conflict-free* policies (i.e., each slot is allocated to at most one station). This conflict-free model is mainly applicable to data communication systems which use a satellite communication channel [12], terrestrial loop circuit [9], or local area networks of computers [1]. It appears that even when collision detection and resolution is cost effective and reliable, it is not worthwhile to allow conflicts when the arrival rates of messages are high. The entire discussion is limited to environments that have a stable enough population so that one may indeed dispose of collision detection and resolution. We further limit our field to *decentralized* policies: no networkwide controls are employed during the transmission operations.

This study is a continuation of Itai and Rosberg [6], where the buffer capacity was set at one (i.e., no queues form) and the throughput of the channel was taken as the cost criterion. It was shown there that the *golden ratio policy* (defined in Section V) achieves a nearly optimal throughput. We consider a more realistic model of a multiple-access system, in which messages arriving at a busy transmitter are not lost but are queued for later transmission. The performance criterion accordingly is now the expected packet delay. By evaluating the performance of the golden ratio policy here and comparing it to (normally unachievable) lower bounds on the delay we show it performs extremely well. This indicates that it is a robust policy as well.

In Section II we formulate the mathematical model and define the set of policies against which we wish to evaluate the golden ratio policy. Then we develop the main analytical tool—a discrete-time single-server queueing model, presented with a load of independently identically distributed (i.i.d.) batches of messages and where the server is allocated to the stations according to a periodic sequence of time slots. The mean queue size at transmitter i, $\mu^{(i)}$ is computed, with the details of the analysis given in the Appendix.

In Section III we first consider a single station and argue that $\mu^{(i)}$ is minimized when the interallocation intervals are equal, and we optimize on the fractions of the channel bandwidths allocated to each station obtaining a lower bound on our objective function. The implementation of this fractional solution gives rise to a "placement problem," as it turns out that the optimal equal interallocation durations cannot be accommodated except in special (degenerate) cases.

In Section IV we present a "random" control policy implementing the fractional allocation distributively and evaluate its closeness to the bound, which turns out to be quite poor. In Section V we present multiplicative hashing using the golden ratio multiplier as the answer to the placement problem and summarize its properties. We then demonstrate the efficacy of the assignment based on this hashing in producing values for the object function that are extremely close to the lower bound (a bound which is

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normally unrealizable). We also discuss there the various components of the difference between the achieved values and their bounds.

II. Model Formulation and Basic Results

Let n be the number of transmission stations and $\{V^{(i)}(t)\}_{t\geq 0}$ $(1 \leq i \leq n)$ be the arrival process of packets during successive time slots at station i. We assume that for each i the $\{V^{(i)}(t)\}_{t\geq 0}$ form a sequence of i.i.d. random variables, with their first two moments and variance denoted by $\lambda^{(i)}$, $\xi^{(i)}$, and $\zeta^{(i)}$, respectively, and that arrival processes to distinct stations are mutually independent.

All packets arriving during a slot join the buffer at slot end. We assume that a transmitter always uses a slot allocated to it if its buffer is not empty at the slot beginning. This is trivially optimal.

Let $X^{(i)}(t)$ denote the number of packets in the buffer of station i at the beginning of slot number t. Let $C^{(i)}$ be the cost per unit time of keeping a packet in station i, $V_T(\pi)$ be the total expected cost until time T of using policy π, and define the long-run average cost

$$\overline{V}(\pi) = \limsup_{T\to\infty} V_T(\pi)/T \tag{2.1}$$

where by definition

$$V_T(\pi) = \sum_{t=1}^{T} E_\pi(C, X(t)) \tag{2.2}$$

with

$$(C, x) = \sum_{i=1}^{n} C^{(i)}x^{(i)}.$$

The expectation in (2.2) is with respect to the probability measure induced by the policy π and the arrival process. From (2.2) it follows that under stationary conditions

$$\overline{V}(\pi) = \sum_{i=1}^{n} C^{(i)}\mu^{(i)} \tag{2.3}$$

where

$$\mu^{(i)} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E_\pi\big(X^{(i)}(t)\big)$$

is the expected queue length (number of buffered packets) at station i at the beginnings of slots under the stationary distribution of the system states.

Remark 2.1: For $C^{(i)} = 1/\lambda^{(i)}$, Little's lemma provides that $\overline{V}(\pi)$ is the long-run average waiting time.

Various approaches to the choice of an optimal policy are discussed in detail in [4]. Here we restrict ourselves to a subset of the distributed policies (see [4]), consisting of conflict-free deterministic allocations of the channel time, in which the sequence of slot assignments is *predetermined*. A common term for this class of policies is weighted time division multiplexing (TDM) (weighted since we do not require that all the stations share the channel equally).

It is an interesting—and so far open—question, just how good is this subset. A further concession to the exigencies of efficient implementation on finite machines is that most of this paper concerns *periodic* TDM policies, also called loop policies.

Definition 2.1: A policy π is a *loop (periodic) policy* if an integer N (its period) exists such that, for all t, the station which is allocated slot t is also allocated slot $t + N$.

Note that under a loop policy the sequence of interallocation times for station i, $d_j^{(i)}$, $j = 0, 1, 2, \cdots$ is periodic. That is, station i is allocated $N^{(i)}$ transmission slots per period, and $d_{N^{(i)}+j}^{(i)} = d_j^{(i)}$ for every j and $\sum_{j=0}^{N^{(i)}-1} d_j^{(i)} = N$ where N is the loop size.

Remark 2.2: It is easy to show, using the generalized Foster criterion (see [11]) that under a loop policy the underlying Markov chain of the queue size at station i is ergodic iff $\lambda^{(i)} < N^{(i)}/N$, $1 \leq i \leq n$.

We turn now to determine the distribution of the buffer occupancies (queue lengths) at each of the n stations. We compute its distribution at the beginning of an arbitrary slot for any ergodic loop policy. This analysis is required to evaluate the average delay per packet under such policies, and, in particular, under the golden ratio policy which is presented in Section V. As a byproduct of this analysis we obtain an explicit form for a lower bound on the cost criterion. The analysis is based on [10].

Since the queues at distinct stations do not interact under such a policy, they are independent, and we may focus on a generic station i. To simplify notation, we omit below the station index. Let N be the loop size and L the number of slots in the loop which are allocated to a given station. Further, let d_j, $0 \leq j < L$ be distances between two successive allocations, and then, $\sum_{j=0}^{L-1} d_j = N$.

During the time from the beginning of the jth transmission slot in the loop to the beginning of the $(j+1)$st, the station is said to be in its jth *phase*, $0 \leq j < L$. During each slot a random number V of packets arrive at the station. Let λ and ξ be its first and second moments, $\zeta = \xi - \lambda^2$ its variance, and $\alpha(z)$ its probability generating function (pgf). Let V^j be the d_j-fold convolution of V and $\alpha_j(z) \equiv \alpha^{d_j}(z)$ the pgf of V^j.

Since the computation is limited to ergodic policies, we may assume that the queue length is stationary (i.e., it does not depend on the time phase 0 in the cycle started). Denote by Y_j the queue length at the beginning of phase j and by $G_j(z)$ its pgf. The proofs of the following two Theorems are given in the Appendix.

Theorem 2.1: a) The pgf of Y_j is given by

$$G_j(z)\big(z^L - \beta(z)\big) = \sum_{k=j}^{j-1} (z-1)p_k z^{[k-j]} \prod_{m=k}^{j-1} \alpha_m(z)$$

$$\beta(z) = \prod_{k=0}^{L-1} \alpha_k(z), \qquad p_k \equiv G_k(0) \tag{2.4}$$

where $[i] = i$ modulo L, and the summation and product indices wraparound from $L-1$ to 0 where needed.

b) The empty queue probabilities are given by

$$p_k \equiv G_k(0) = (L - \lambda N)\eta_k, \qquad k = 0, 1, \cdots, L-1,$$

and the vector $\eta = (\eta_1, \eta_2, \cdots, \eta_{L-1})$ is defined in the Appendix, through (A.9).

Although Theorem 2.1 provides us with the *distribution* of Y_j, all that we need in (2.3) is the expected queue length μ at the beginning of an arbitrary slot. This is given via the following theorem.

Theorem 2.2: The mean queue length at an arbitrary slot beginning is

$$\mu = \frac{(\zeta+\lambda)N + \sum_{j=0}^{L-1} \lambda d_j\left(\lambda d_j - 2(1-p_j)\right) - 2\sum_{j=0}^{L-1}(1-\lambda d_j)\sum_{k=0}^{j-1}(\lambda d_k - (1-p_k))}{2(L-\lambda N)} + \frac{1}{N}\sum_{j=1}^{L-1} d_j \sum_{k=0}^{j-1}(\lambda d_k - (1-p_k)) + \frac{1}{N}\sum_{j=0}^{L-1} d_j\left[\frac{\lambda(d_j-1)}{2} - (1-p_j)\right]. \tag{2.5}$$

For the ensemble of n stations denote by $\mu^{(i)}$ the expected queue length at station i at the beginning of an arbitrary slot. For a given loop policy, let $d_j^{(i)}, 0 \le j < N^{(i)}$ be the distances between successive allocations to station i. Replacing λ, ζ, L, d_j, and p_k in Theorem 2.2 by $\lambda^{(i)}$, $\zeta^{(i)}$, $N^{(i)}$, $d_j^{(i)}$, and $p_k^{(i)}$, we obtain an expression for $\mu^{(i)}$.

Theorem 2.3: Under every loop policy π

$$\bar{V}(\pi) = \sum_{i=1}^{n} C^{(i)}\mu^{(i)}$$

where $\mu^{(i)}$ is given in Theorem 2.2 with the above substitutions.

In the next section we shall use these results to develop a lower bound to the value of $\bar{V}(\pi)$, for all weighted TDM policies.

III. A Lower Bound on $\bar{V}(\pi)$

In this section we bound $\bar{V}(\pi)$ from below for every weighted TDM policy π, not necessarily periodic. Let π be a weighted TDM policy. From (2.3) follows $\bar{V}(\pi) = \sum_{i=1}^{n} C^{(i)}\mu^{(i)}$, where for loop policies the $\mu^{(i)}$ are given in (2.5), substituted for station i. We proceed as follows to find the bound.

1) For each $\mu^{(i)}$ we independently obtain a lower bound, based on minimizing it under the constraint that the average interallocation distance of the station (its mean phase length, in terms of Section II) is equal to a specified value $d^{(i)}$. These minima are denoted by $B^{(i)}$.

2) Then we consider the ensemble of n stations and solve the minimization problem

$$\min_{\{d^{(i)}\}} \sum_{i=1}^{n} C^{(i)}B^{(i)} \qquad \text{such that } \sum_{i=1}^{n}\frac{1}{d^{(i)}} = 1.$$

Let i be a given station and $d^{(i)}$ be a given real number larger than one. Further, let $\pi(d^{(i)})$ be any weighted TDM policy for which the sequence of interallocation times to station i, $d_j^{(i)}$, $j = 0, 1, 2, \cdots$ satisfies

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} d_j^{(i)} \ge d^{(i)}. \tag{3.1}$$

Viewing the interallocation times as real numbers, for the time being, rather than integers, the optimization $\min_{\{\pi(d^{(i)})\}} \mu^{(i)}$ subject to satisfying the relation (3.1) will produce a lower bound for the $\mu^{(i)}$, among all weighted TDM policies. This leads to the following.

Theorem 3.1 (Local Optimum): $\mu^{(i)}$ is minimized when

$$d_j^{(i)} = d^{(i)}.$$

Remark 3.1: 1) The optimization problem here is similar to the problem analyzed in [2] and later generalized, as well as a much simpler proof in [5]. The results in [5] do not immediately apply to our problem since two of the assumptions made there are violated in our context. First, the process of interarrival times of single messages in our model is not stationary in the sense that was used in [5]. This is due to the possible arrivals of batches here. An alternative way of phrasing the difference is that we use a different type of time indexing: "services" of the channel start on a discrete set of epochs. Nevertheless, a proof proceeding along the same lines as in [5], *mutatis mutandis*, can be carried out (at some cost in increased complexity).

2) An alternative approach to proving Theorem 3.1 for *loop policies* only consists in minimizing μ from (2.5) subject to the constraints on the interallocation distances d_j and the boundary probabilities p_k (see (A.15)). Forming the implied Lagrangian and equating its derivatives to zero provides the same result. This is fully done in [4].

3) For a recent discussion of these issues of delay minimization see [13].

Remark 3.2: When $d^{(i)}$ is not integral, an achievable minimum in Theorem 3.1 is obtained by "nearly uniform" determination of $\{d_j^{(i)}\}$ as is shown by Hajek [3]. This allocation scheme is as follows.

Slot t is allocated to station i iff $\lfloor (t+1)/d^{(i)} \rfloor - \lfloor t/d^{(i)} \rfloor = 1$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. Performing these calculations on a finite precision machine takes special care. Unless $d^{(i)}$ is a rational number this will not result in a periodic policy. Also, for reasons discussed in Section V this assignment will not be feasible for more than two stations.

Corollary 3.1: Under every weighted TDM policy π, which satisfies (3.1),

$$\mu^{(i)} \geq \frac{d^{(i)}\zeta^{(i)}}{2(1-\rho_i)} - \frac{\lambda^{(i)}}{2} \tag{3.2}$$

where $\rho_i = \lambda^{(i)}d^{(i)}$ and $\zeta^{(i)} = \xi^{(i)} - (\lambda^{(i)})^2$.

Proof: The corollary is immediate by replacing d_j with $d^{(i)}$ in (2.6) and using Theorem 3.1.

So far in this section we have minimized the contribution of a *single* generic transmitter to the object function. Now consider the ensemble of n stations, and let $\pi(d)$ be an ergodic policy with the average distance between the allocations it grants station i equaling $d^{(i)}$, $1 \leq i \leq n$. From corollary (3.1) and Theorem 2.1 we have the lower bound.

Theorem 3.2: For every weighted TDM policy $\pi(d)$, which satisfies (3.1) for $1 \leq i \leq n$,

$$\overline{V}(\pi(d)) \geq \sum_{i=1}^{n} C^{(i)}\left[\frac{d^{(i)}\zeta^{(i)}}{2(1-\rho_i)} - \frac{\lambda^{(i)}}{2}\right]. \tag{3.3}$$

The theorem introduced below gives a further characterization of the lower bound to $\overline{V}(\pi)$. Let

$$x^{(i)} = \frac{1}{d^{(i)}}, \tag{3.4}$$

so that $x^{(i)}$ is the long-run fraction of slots allocated to station i, and $\sum_{i=1}^{n} x^{(i)} = 1$. Thus from (3.4) and Theorem 3.2 the optimal solution to the following optimization problem would be a lower bound to the values $\overline{V}(\pi(d))$ over all d that produce stable systems:

$$\min_{\{x^{(i)}\}} \left\{ \sum_{i=1}^{n} \frac{C^{(i)}\zeta^{(i)}}{2(x^{(i)} - \lambda^{(i)})} - \frac{1}{2}\sum_{i=1}^{n} C^{(i)}\lambda^{(i)} \right\}$$

$$\text{such that}^1 \sum_{i=1}^{n} x^{(i)} = 1, \qquad x^{(i)} > \lambda^{(i)}. \tag{3.5}$$

The function that is being minimized here is just the right-hand side of (3.3) with $\lambda^{(i)}/x^{(i)}$ substituted for ρ_i.

The problem posed in (3.5) is familiar in other contexts as well. Adopting the solution presented in [7, pp. 329ff] (see also [4] for a detailed treatment), we have the following.

Theorem 3.3: For every weighted TDM policy π

$$\overline{V}(\pi) \geq \frac{1}{2(1-\lambda)}\left(\sum_{i=1}^{n} \left(C^{(i)}\zeta^{(i)}\right)^{1/2}\right)^2 - \frac{1}{2}\sum_{i=1}^{n} C^{(i)}\lambda^{(i)},$$

$$\lambda \equiv \sum_{i=1}^{n} \lambda^{(i)}$$

where the right-hand side is obtained from (3.3) and (3.4) using the optimal allocation fractions:

$$x^{(i)} = \lambda^{(i)} + (1-\lambda)\frac{\left(C^{(i)}\zeta^{(i)}\right)^{1/2}}{\sum_{j=1}^{n}\left(C^{(j)}\zeta^{(j)}\right)^{1/2}}. \tag{3.6}$$

Corollary 3.2: If $\lambda^{(i)} = \lambda^{(1)}$ and $C^{(i)}\zeta^{(i)} = C^{(1)}\zeta^{(1)}$, $2 \leq i \leq n$, then the *round robin* (RR) policy is optimal. This policy assigns to station i slots i (modulo n) $-$ 1.

Proof: From (3.6) we have $x^{(i)} = 1/n$. Since under the RR policy the fractional allocations are $1/n$, and the allocations to each station are equidistant with $d = n$, then by Corollary 3.1 the lower bound is obtained.

Consider the $x^{(i)}$ in (3.6), and let $d^{(i)} = 1/x^{(i)}$. An equally spaced loop policy (as in Corollary 3.2) for all the stations is almost never feasible. The infeasibility springs from two sources:

1) usually the $x^{(i)}$ are irrational, thus a finite loop is precluded;
2) even when the $x^{(i)}$ are rational, the implied phase lengths for different stations will rarely agree (see Example 3.1, to follow).

Therefore, we try to approximate the optimal solution by policies which permit station i to transmit $N^{(i)} \approx x^{(i)}N$ times, at nearly equally spaced slots in a loop of size N. Thus we first calculate the optimal $x^{(i)}$ according to (3.6) and then consider policies that give $N^{(i)}$ permissions to station i. These policies will be close to optimal only if the permissions to each station are fairly regularly spaced. Thus we are confronted with the following *placement problem*.

Given

$$N^{(1)}, N^{(2)}, \cdots, N^{(n)}, \sum_{i=1}^{n} N^{(i)} = N,$$

place the permissions of each station thus that

$$\lfloor N/N^{(i)} \rfloor \leq \underline{d}^{(i)} \leq \bar{d}^{(i)} \leq \lceil N/N^{(i)} \rceil$$

where $\underline{d}^{(i)}$ and $\bar{d}^{(i)}$ are the minimum and the maximum distances between successive permissions to station i. This problem does not always have a solution.

Example 3.1: When $n = 3$, $N = 6$, $N^{(1)} = 1$, $N^{(2)} = 2$, and $N^{(3)} = 3$, then ideally station 1 should be given permission once in the loop, station 2 every half-loop, and station 3 every other slot. This is infeasible.

Note that this incompatibility does not spring from the irrationality of the $x^{(i)}$; it would exist even if the optimal allocation fractions calculated from (3.6) for the above example happened to be 1/6, 1/3, and 1/2. The same infeasibility will hold for *any* N.

To bypass this intractable placement problem, we introduce and analyze in the next two sections two realizable control policies. The first one is a "random" control policy, and the second is a special weighted TDM policy, the golden ratio loop policy.

[1]The form of the constraint assures the adoption only of allocation patterns that make the buffer occupancies ergodic.

IV. A Random Control Policy

The purpose of this short section is to demonstrate the merit of a deterministic control policy compared with a random one. We use a specific example to quantify this merit. Consider the channel capacity allocation as given in (3.6) with all $C^{(i)}$ equal to one:

$$x^{(i)} = \lambda^{(i)} + (1 - \lambda)\frac{\sqrt{\zeta^{(i)}}}{\sum_{j=1}^{n} \sqrt{\zeta^{(j)}}}. \tag{4.1}$$

Definition 4.1: Let π_R be the conflict-free policy which at every slot t, $t = 1, 2, \cdots$ permits station i to transmit with probability $x^{(i)}$, $i = 1, 2, \cdots, n$.

The behavior of this system under π_R would be indistinguishable from the one observed when station i is allowed to transmit when a common pseudorandom generator (calibrated to produce numbers uniformly and independently distributed on $[0,1]$) outputs a value x that satisfies

$$\sum_{j=0}^{i-1} x^{(j)} \leq x < \sum_{j=0}^{i} x^{(j)}.$$

Remark 4.1: "Coordinated randomness" can be implemented distributively if, for example, all the stations are synchronized with respect to the time origin and are using identical pseudorandom number generators. Note that time synchronization is required for *any* distributed TDM policy.

To evaluate π_R, we compute the mean stationary buffer occupancy at station i, $i = 1, 2, \cdots, n$ and compare it to the lower bound obtained in Corollary 3.1,

$$B^{(i)} = \frac{\zeta^{(i)}}{2(x^{(i)} - \lambda^{(i)})} - \frac{\lambda^{(i)}}{2} \tag{4.2}$$

where $d^{(i)}$ was replaced by $1/x^{(i)}$. Under π_R station i is a GI/Geom./1 queueing system and through a treatment isomorphic to an M/G/1 queue (because successive slots receive independent input) one obtains the mean stationary buffer occupancy at station i, under π_R as

$$\mu^{(i)}(\pi_R) = \frac{\zeta^{(i)} + \lambda^{(i)} - (\lambda^{(i)})^2}{2(x^{(i)} - \lambda^{(i)})}. \tag{4.3}$$

Consider first the fractional allocations as prescribed by (4.1): since (4.1) and (4.2) imply $\lambda^{(i)} < x^{(i)} < \lambda^{(i)} + 1 - \lambda$, it follows from (4.2) and (4.3) that

$$1 + \frac{\lambda^{(i)}(1 - \lambda^{(i)})}{\zeta^{(i)}} \leq \frac{\mu^{(i)}(\pi_R)}{B^{(i)}} \leq \frac{\zeta^{(i)} + \lambda^{(i)}(1 - \lambda^{(i)})}{\zeta^{(i)} - \lambda^{(i)}(1 - \lambda)}.$$

For illustration, if the number of arrivals per slot has the Poisson distribution, then $\zeta^{(i)}$ is $\lambda^{(i)}$ and

$$2 - \lambda^{(i)} \leq \frac{\mu^{(i)}(\pi_R)}{B^{(i)}} \leq \frac{2 - \lambda^{(i)}}{\lambda}, \tag{4.4}$$

whereby one also gets an idea about the closeness of these bounds, remembering that λ is bounded above by one, and $\lambda^{(i)}$ is a fraction thereof, typically λ/n.

A referee has pointed out that the comparison (4.4) is "unfair" since the allocations $x^{(i)}$ that are optimal for the TDM policy need not be such for π_R. Indeed, since the $\mu^{(i)}(\pi_R)$ have the same functional dependence on $x^{(i)}$ as displayed in (3.5), we can use (3.6) to find the optimal allocations for $\mu^{(i)}(\pi_R)$:

$$x_R^{(i)} = \lambda^{(i)} + (1 - \lambda)\frac{\sqrt{\zeta^{(i)} + \lambda^{(i)}(1 - \lambda^{(i)})}}{\sum_{j=1}^{n} \sqrt{\zeta^{(j)} + \lambda^{(j)}(1 - \lambda^{(j)})}}$$

which do differ from $x^{(i)}$. However, comparing $B^{(i)}$ and $\mu^{(i)}(\pi_R)$ under the optimal allocation for each, the picture does not change much:

$$\frac{\mu^{(i)}(\pi_R)}{B^{(i)}} = \frac{\sqrt{\zeta^{(i)} + \lambda^{(i)}(1 - \lambda^{(i)})} \sum_{j=1}^{n} \sqrt{\zeta^{(j)} + \lambda^{(j)}(1 - \lambda^{(j)})}}{\left[\sum_{j=1}^{n} \sqrt{\zeta^{(j)}}\right] \sqrt{\zeta^{(i)}} - \lambda^{(i)}(1 - \lambda)}.$$

For an illustration similar to (4.4), assume Poisson arrivals as before and all $\lambda^{(i)}$ equal; then

$$\frac{\mu^{(i)}(\pi_R)}{B^{(i)}} = \frac{2 - \lambda^{(i)}}{\lambda^{(i)} + 1 - 1/n},$$

which is not much of an improvement, as the bounds (4.4) still hold. As we shall see in the next section, we can do much better.

V. The Golden Ratio Control Policy

Let $x^{(i)} > 0$, $i = 1, 2, \cdots, n$, $\sum_{i=1}^{n} x^{(i)} = 1$ be the desirable fractions of permissions to each of the stations, as given by (3.6). Also, let F_k be the kth Fibonacci number and $N_k^{(i)}$, $i = 1, 2, \cdots, n$ be integers such that

$$\lfloor x^{(i)} F_k \rfloor \leq N_k^{(i)} \leq \lceil x^{(i)} F_k \rceil \qquad \sum_{i=1}^{n} N_k^{(i)} = F_k. \tag{5.1}$$

These relations provide

$$\lim_{k \to \infty} \frac{N_k^{(i)}}{F_k} = x^{(i)}. \tag{5.2}$$

For each k, the golden ratio policy assigns $N_k^{(i)}$ slots to station i and attempts to distribute the permissions uniformly over a loop of size F_k. (The analysis of Section III implies that it is optimal to distribute the permissions regularly.)

Open address hashing confronts a similar problem: to distribute keys uniformly over a hash table. The uniformity of the distribution depends on the hash function. It has been shown that multiplicative hashing with the golden ratio multiplicand $\phi^{-1} = (\sqrt{5} - 1)/2 \approx 0.6180339887$ distributes the keys most uniformly [8]. The golden ratio policy applies some of these results. Fibonacci numbers are related to the golden ratio ϕ^{-1} by

$$F_k = \frac{\phi^k - (1-\phi)^k}{\sqrt{5}}.$$

Let $\mathrm{frac}(y) = y - \lfloor y \rfloor$, $a_j = \mathrm{frac}(j\phi^{-1})$, and $A_N = \{a_j | j = 0, \cdots, N-1\}$. The tth smallest point of A_{F_k} is associated with the tth slot of the loop.

Definition 5.1: The golden ratio policy $\pi_{\mathrm{GR}(k)}$ is a loop policy which uses a period of F_k slots and assigns to station i the slots corresponding to the $N_k^{(i)}$ points

$$\left\{ a_j \middle| \sum_{m=1}^{i-1} N_k^{(m)} \le j < \sum_{m=1}^{i} N_k^{(m)} \right\}.$$

It will be convenient to identify the points 0 and 1, and thus the points a_j are distributed over a unit circle.

Example 5.1: Suppose $n = 3$, $x^{(1)} = 1/2 \pm \epsilon_1$, $x^{(2)} = 3/8 \pm \epsilon_2$, $x^{(3)} = 1/8 \pm \epsilon_3$, where $\epsilon_i > 0$ are arbitrarily small and $x^{(1)} + x^{(2)} + x^{(3)} = 1$. Taking $F_6 = 8$, $N_6^{(1)} = 4$, $N_6^{(2)} = 3$, and $N_6^{(3)} = 1$, $\pi_{\mathrm{GR}(6)}$ assigns to station 1 the slots corresponding to 0, ϕ^{-1}, $\mathrm{frac}(2\phi^{-1})$, and $\mathrm{frac}(3\phi^{-1})$; to station 2 the slots corresponding to $\mathrm{frac}(4\phi^{-1})$, $\mathrm{frac}(5\phi^{-1})$, and $\mathrm{frac}(6\phi^{-1})$; and to station 3 the point corresponding to $\mathrm{frac}(7\phi^{-1})$. Thus the loop policy keeps giving permission to the stations in the following cyclic order: "1, 2, 1, 3, 2, 1, 2, 1."

In the following two theorems, which are proved in [6, sec. 5], we give the regularity characteristics of $\pi_{\mathrm{GR}(k)}$. Let j_i satisfy $\phi^{-j_i} < x^{(i)} \le \phi^{-j_i+1}$, $1 \le i \le n$.

Theorem 5.1: For each station i with $N_k^{(i)} = F_{k_i} + s_k^{(i)}$, $0 \le s_k^{(i)} < F_{k_i - 1}$ where $k_i = k - j_i$, there are distances of at most three values between consecutive allocations:

$$s_k^{(i)} \text{ occurrences of distance } F_{k-k_i}$$

$$F_{k_i-2} + s_k^{(i)} \text{ occurrences of distance } F_{k-k_i+1}$$

$$F_{k_i-1} - s_k^{(i)} \text{ occurrences of distance } F_{k-k_i+2}.$$

Remark 5.1: It is also shown in [6] that the different distances are uniformly mixed.

Remark 5.2: If a loop of a length which is not a Fibonacci number is used for the foregoing allocation, the results are similar in regularity but more values of distances are generated.

Theorem 5.2: For each station i with $N_k^{(i)} = F_{k_i} + s_k^{(i)}$, $0 \le s_k^{(i)} < F_{k_i-1}$ allocations in a loop of size $N = F_k$, and k being sufficiently large, the following proportions of distances are generated:

$$1 - \frac{\phi^{-j_i}}{x^{(i)}} \text{ of distances } F_{j_i}$$

$$\frac{\phi^{-j_i-2} + \phi^{-j_i}}{x^{(i)}} - 1 \text{ of distance } F_{j_i+1}$$

$$1 - \frac{\phi^{-j_i-2}}{x^{(i)}} \text{ of distance } F_{j_i+2}.$$

The policy $\pi_{\mathrm{GR}(k)}$ defines the distances $d_j^{(i)}$, $j = 0, 1, \cdots, N_k^{(i)} - 1$, $1 \le i \le n$. Now $\mu^{(i)}$ and $\overline{V}(\pi_{\mathrm{GR}(k)})$ can be computed using Theorems 2.2 and 2.3. To evaluate the quality of $\pi_{\mathrm{GR}(k)}$, we need to compute $E[\overline{V}/\overline{V}(\pi_{\mathrm{GR}(k)})]$, where $\overline{V}$ is the value function, the overall optimum, as a function of the number of transmitting stations and the size of the loop used by $\pi_{\mathrm{GR}(k)}$, where the expectation is taken in some suitable sense over the range of admissible arrival processes. This we cannot do, first because $\overline{V}$ is not known, and secondly, there is no satisfactory measure over the set of arrival processes.

To obviate the first difficulty we use instead of $\overline{V}$ the *lower* bound (for TDM policies) provided by Theorem 3.3. The second one is in a sense circumvented by estimating the ratio at an arbitrarily selected set of points: only Poisson arrival processes were considered, and two load levels were chosen, $\lambda = 0.3$ and 0.9. To obtain the values of $\lambda^{(i)}$, we sampled values from a gamma distribution, with parameters computed to produce coefficient of variation over the collection of $\lambda^{(i)}$, to be denoted by γ, at the six values 0.1, 0.25, 0.5, 1, 2, and 10. This plan was adopted on the hypothesis that variation among the $\lambda^{(i)}$ is likely to be an important determinant of the performance of the channel (under any policy). Support for this hypothesis may be found in the expression for the bound on $\overline{V}$ in Theorem 3.3; specializing for Poisson arrival processes, where $\zeta^{(i)} = \lambda^{(i)}$, then $\Sigma(\lambda^{(i)})^{1/2}$ increases (in expectation) as the variation of the population from which the $\lambda^{(i)}$ are drawn decreases. Computations were performed for 5, 15, and 30 transmitting stations, and loop lengths were fixed at 89, 233, and 610 slots (for five stations 55 slots were used rather than 610); note that these are all Fibonacci numbers. The cost coefficients $C^{(i)}$ were taken all equal to one. Finally, we observe that $\overline{V}(\pi_{\mathrm{GR}(k)})$ fails to achieve $\overline{V}$ for two distinct reasons.

1) It uses a finite loop, hence the optimal capacities as given by (3.6) cannot usually be assigned, due to integrality constraints.
2) The policy $\pi_{\mathrm{GR}(k)}$ does not normally produce for a given loop and $N^{(i)}$ the best possible placement. Indeed, in the special case where the $\lambda^{(i)}$ are close enough and n divides N, $N^{(i)} = N/n$ is clearly the optimal capacity assignment and π_{RR}, the round-robin policy, is the optimal placement; still, $\pi_{\mathrm{GR}(k)}$ when $F_k > n$ would use interallocation distances of (usually) three distinct values. In the case when $F_k = n$ we of course get $\pi_{\mathrm{GR}(k)} = \pi_{\mathrm{RR}}$.

To estimate the contribution of the first factor, we use the bound of Theorem 3.3 both with the optimal allocation from (3.6) and the actually achieved capacities.[2] We checked at a few points the difference between $\pi_{\mathrm{GR}(k)}$ and

[2] The latter were obtained by first rounding $x^{(i)}N$ and, if these did not sum to N, by adding or deleting slots from allocations where the resulting difference $|x^{(i)}N - N^{(i)}|$ is then the smallest. As a result, the relation (5.1) is not invariably satisfied. Again, this is not necessarily optimal, but appears to have caused a loss that is vastly dominated by the others.

TABLE I
PERFORMANCE RATIOS FOR DECREASING ARRIVAL RATE VARIATION

γ	λ	n=5			n=15			n=30		
N	:	89	233	55	89	233	610	89	233	610
10.0	0.3	.9221	.9178	.9152	.8862	.9181	.9226	.7838	.9026	.9237
		.9266	.9182	.9235	.9300	.9264	.9241	.9482	.9324	.9309
	0.9	.8807	.9696	.8446	.6156	.6859	.0000	.0000	.3897	.7816
		.9888	.9850	.9903	.9934	.9908	.0000	.0000	.9956	.9898
2.0	0.3	.9151	.9179	.9206	.9129	.9279	.9297	.8696	.9225	.9250
		.9165	.9181	.9235	.9338	.9295	.9300	.9531	.9337	.9263
	0.9	.9040	.9769	.7573	.2914	.8678	.9732	.0000	.4989	.9145
		.9882	.9851	.9910	.9974	.9886	.9845	.0000	.9950	.9868
1.0	0.3	.9219	.9184	.9159	.9326	.9261	.9254	.9232	.9286	.9271
		.9229	.9186	.9183	.9393	.9270	.9256	.9566	.9333	.9277
	0.9	.9118	.9746	.8416	.5005	.9104	.9669	.0000	.6796	.9443
		.9872	.9840	.9898	.9950	.9882	.9833	.0000	.9931	.9879
0.5	0.3	.9222	.9180	.9187	.9320	.9297	.9255	.9369	.9332	.9291
		.9230	.9181	.9205	.9376	.9305	.9256	.9581	.9361	.9296
	0.9	.9532	.9794	.8776	.5724	.9458	.9840	.0000	.7803	.9657
		.9868	.9844	.9891	.9944	.9880	.9850	.0000	.9917	.9868
0.25	0.3	.9175	.9151	.9174	.9297	.9268	.9252	.9450	.9343	.9274
		.9181	.9152	.9193	.9346	.9276	.9253	.9637	.9374	.9278
	0.9	.9614	.9806	.9145	.6169	.9470	.9808	.0000	.7819	.9667
		.9873	.9850	.9885	.9941	.9876	.9855	.0000	.9916	.9873
0.1	0.3	.9077	.9057	.9062	.9217	.9194	.9190	.9603	.9488	.9422
		.9082	.9058	.9079	.9266	.9200	.9191	.9718	.9519	.9426
	0.9	.9557	.9797	.8725	.5753	.9448	.9794	.0000	.7979	.9674
		.9864	.9838	.9880	.9942	.9867	.9844	.0000	.9929	.9880

π_{RR}, where the latter was indeed optimal, and the values appear to be in line with the others, which rather strengthens the evidence for the following conclusions.

Denote the bound with the optimal capacities by V_1 and the one with the actually achieved capacities by V_2. Let $r_i = V_i/\overline{V}(\pi_{GR(k)})$, $i = 1, 2$. In Table I appear the values computed for r_i. In each case the upper value is r_1 and the lower one is r_2. Each point is averaged from ten samples of $\lambda^{(i)}$. The following conclusions may be drawn.

1) r_2 is bounded from below by 0.9, i.e., the placement produced by $\pi_{GR(k)}$ "costs" less than ten percent and often much less.

2) The achieved "rational" allocations harm the performance very little unless "difficult conditions" transpire (some of the following was gleaned from more detailed printouts):

a) λ is high, but stations with very low $\lambda^{(i)}$ still get one slot (even if their optimal allocation is much less than $1/N$). This "pushes" other stations to values of $N\lambda^{(i)}/N^{(i)}$ very close to one. Actually, the zeros in the table are all for $\lambda = 0.9$, and they correspond with one exception to cases where in all the samples some of the stations were reduced to instability in just such a way. (The exception for $n = 15$, $\gamma = 10$, and $N = 610$ arose because in all samples some stations were assigned too many slots for the computation to take reasonable time. We allowed up to 120 slots per station, and at this point in the sample space the $\lambda^{(i)}$ vary widely.) As N increases the above effect diminishes.

b) The ratio N/n is too low for $x^{(i)}N$ to be well approximated. This is more harmful the higher is λ (again, because overallocation for one station would bring others to too high utilization levels); it happens more often when the coefficient of variation of $\lambda^{(i)}$ is large (the first phenomenon is then also more pronounced).

3) The extent of the variation among $\lambda^{(i)}$ did not affect the placement algorithm at all, but as noted earlier it occasionally impaired the capacity assignment mechanism.

The computations were robust, and no precision problems were apparent. They required, however, substantial computing time, most of it in the iteration outlined in Remark A.1. The use of an arrival process of smaller support for the distribution of V (such as the Bernoulli distribution) would reduce it considerably: for the rates and loops we used, we estimate an order of magnitude reduction.

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APPENDIX

In this Appendix we provide the proofs for Theorems 2.1 and 2.2. The proofs are essentially computational and use the notation established in Section II. We draw on methods developed by Neuts in [10].

Proof of Theorem 2.1

The evolution of the queue length at successive phase beginnings is governed by

$$Y_{[j+1]} = Y_j - I(Y_j > 0) + V^j \tag{A.1}$$

where $[j+1]$ is $(j+1)$ modulo L. Thus $G_{j+1}(z)$ is related to $G_j(z)$ through the operator T_j:

$$G_{j+1}(z) = T_j[G_j(z)] \equiv \alpha_j(z)\left[G_j(0) + \frac{1}{z}(G_j(z) - G_j(0))\right]. \tag{A.2}$$

By iterating (A.2) L times, we have

$$G_j(z)(z^L - \beta(z)) = \sum_{k=j}^{j-1} (z-1) p_k z^{[k-j]} \prod_{m=k}^{j-1} \alpha_m(z) \quad \text{(A.3)}$$

where

$$\beta(z) = \prod_{k=0}^{L-1} \alpha_k(z) \qquad p_k \equiv G_k(0)$$

and the summation and product indices wraparound from $L-1$ to 0 where needed. This establishes part a) of Theorem 2.1. In this way, all the queue length distributions are determined up to the boundary probabilities p_k, $k = 0, \cdots, L-1$. We now turn to evaluate them.

To compute the boundary probabilities, we use the following device.

- Define a random matrix C_{ij}, $0 \le i, j \le L-1$, of the down level crossing periods (dlcp's).
- $\Pr(C_{ij} = r) = \Pr$ (starting with the allocation mechanism at the beginning of phase i and k packets in the queue, the queue length reaches $k-1$ for the first time after r allocations to the station and this happens at the beginning of phase j).

We first evaluate the distribution of the C_{ij} and then show how they are related to the boundary probabilities p_j.

1) The Distribution of C_{ij}: From the definitions and introducing the notation $q_j(k) = \Pr(V^j = k)$,

$$\Pr(C_{ij} = 1) = q_i(0)\delta_{j,[i+1]}, \qquad (q_i(0) = \alpha_i(0)) \qquad \text{(A.4i)}$$

$$\Pr(C_{ij} = r) = \sum_{k=1}^{r-1} q_i(k) \Pr\left(C^{*k}_{[i+1],j} = r-1\right), \qquad r > 1 \qquad \text{(A.4ii)}$$

where δ_{ij} is the Kronecker delta and C^{*k} is the k-fold convolution of C.

Define the generating function

$$\gamma_{ij}(z) = \sum_{r \ge 1} \Pr(C_{ij} = r) z^r, \quad \text{and} \quad \gamma(z) = (\gamma_{ij}(z)).$$

Then from (A.4),

$$\gamma_{ij}(z) = z q_i(0)\delta_{[i+1],j} + \sum_{k\ge 1} q_i(k) z \gamma^k_{[i+1],j}(z) \quad \text{(A.5)}$$

where $\gamma^k_{i,j}(z)$ are the elements of $\gamma^k(z)$, the matrix $\gamma(z)$ raised to the kth power. In matrix form (A.5) becomes

$$\gamma(z) = z \sum_{k \ge 0} H(k) \gamma^k(z) \qquad \text{(A.6)}$$

with the matrix $H(k)$ given by

$$H(k) = \begin{bmatrix} 0 & q_0(k) & 0 & \cdots & 0 \\ 0 & 0 & q_1(k) & \cdots & \cdot \\ \cdot & \cdot & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 0 & \cdot & \cdot & \cdots & q_{L-2}(k) \\ q_{L-1}(k) & 0 & \cdot & \cdots & 0 \end{bmatrix}, \qquad k \ge 0. \quad \text{(A.7)}$$

Defining $\gamma \equiv \gamma(1)$, the matrix γ satisfies

$$\gamma = \sum_{k \ge 0} H(k) \gamma^k. \qquad \text{(A.8)}$$

It is easy to verify that γ is stochastic. Let $\eta = (\eta_0, \cdots, \eta_{L-1})$ be the stationary probability vector of γ, that is

$$\eta = \eta\gamma, \qquad \sum_{i=0}^{L-1} \eta_i = 1. \qquad \text{(A.9)}$$

2) The Boundary Probabilities: Consider the process that assumes the value of the phase number at those instances where the *queue is empty* at beginnings of phases. Clearly, this process is a first-order Markov chain.

Let $A = (a_{ij})$ be its transition probability matrix; that is, $a_{ij} = \Pr$ (given that $Y_i = 0$, the next time the queue is empty happens at the beginning of phase j). By conditioning on the value of V^i, we have

$$\begin{aligned} a_{ij} &= q_i(0)\delta_{[i+1]j} \\ &\quad + \sum_{k \ge 1} q_i(k) \times \Pr\{\text{given that } Y_{[i+1]} = k, \text{ the queue} \\ &\qquad \text{will first assume the value 0 at phase } j\}. \qquad \text{(A.10)} \end{aligned}$$

The probability of the event in braces is $\gamma^k_{[i+1],j}$.

From (A.5) and (A.10)

$$A = \gamma \equiv \gamma(1). \qquad \text{(A.11)}$$

Thus η_j (the jth element of η, γ's stationary probability vector) is the stationary probability of being at the beginning of phase j, given that the queue is empty. Also,

$$\begin{aligned} p_j &= \Pr(\text{the queue is empty} \\ &\qquad | \text{the system is at the beginning of phase } j) \\ &= \Pr(\text{the system is at the beginning of phase } j \\ &\qquad | \text{the queue is empty}) \\ &\qquad \cdot \Pr(\text{queue is empty}) \cdot \frac{1}{\Pr(\text{the system is at phase } j)}. \qquad \text{(A.12)} \end{aligned}$$

Taking the expectation of (A.1) with respect to the stationary distribution and then iterating it L times gives

$$\Pr(\text{the queue is empty at phase beginning}) = 1 - \lambda N/L. \qquad \text{(A.13)}$$

Since the station "visits" all phases with equal frequency,

$$\Pr(\text{the system is at phase } j) = \frac{1}{L}. \qquad \text{(A.14)}$$

Now part b) of Theorem 2.1 follows from (A.11)–(A.14).

Remark A.1: The matrix γ can be computed by the following successive approximations:

$$\gamma_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdot \\ \cdot & \cdot & 0 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 0 & \cdot & \cdot & \cdots & 1 \\ 1 & 0 & \cdot & \cdots & 0 \end{bmatrix}$$

and

$$\gamma_{j+1} = \sum_{k \ge 0} H(k) \gamma_j^k, \qquad j \ge 0.$$

Since γ is stochastic and $H(k)$ a probability function, $\gamma_j \to \gamma$. The convergence is geometric, at a rate that depends on the largest eigenvalue of γ which is less than one.

Proof of Theorem 2.2

Now $\overline{Y}_j$, the first moment of Y_j, can be obtained by differentiating (A.3) at $z = 1$, and also directly from (A.1). By taking the expectation of (A.1) and iterating the resulting equation L times, it follows from the definitions that

$$\sum_{k=0}^{L-1} (1 - p_k) = \lambda N. \tag{A.15}$$

By squaring (A.1), then taking the expectation with respect to the stationary distribution, iterating it L times, and using (A.1) and (A.15), we obtain

$$\overline{Y}_0 = \frac{(\xi + \lambda - \lambda^2) N + \sum_{j=0}^{L-1} \lambda d_j(\lambda d_j - 2(1 - p_j)) - 2\sum_{j=0}^{L-1} (1 - \lambda d_j) \sum_{k=0}^{j-1} (\lambda d_k - (1 - p_k))}{2(L - \lambda N)} \tag{A.16}$$

where the empty sum vanishes, and

$$\overline{Y}_j = \overline{Y}_0 + \sum_{k=0}^{j-1} (\lambda d_k - (1 - p_k)), \qquad 1 \le j < L. \tag{A.17}$$

Let Y be station i's buffer occupancy at the beginning of an arbitrary slot (not necessarily one which is allocated to station i) and μ its expectation. Also let $Y_j(k)$ be the buffer occupancy, k slots after the beginning of phase j:

$$Y_j(k) = Y_j - I(Y_j > 0) + V^k, \qquad k = 1, 2, \cdots, d_j. \tag{A.18}$$

Clearly, $Y_j(d_j) = Y_{[j+1]}$.

Since the loop contains N slots "visited" uniformly,

$$\begin{aligned}\mu &= \frac{1}{N} \sum_{j=0}^{L-1} \sum_{k=0}^{d_j-1} \overline{Y}_j(k) \\ &= \frac{1}{N} \sum_{j=0}^{L-1} d_j \left[\overline{Y}_j - (1 - p_j) + \frac{1}{2}\lambda(d_j - 1) \right]. \end{aligned} \tag{A.19}$$

From (A.16) through (A.19) we have Theorem 2.2: the mean buffer content at a random slot beginning is given by

$$\begin{aligned}\mu = {} & \frac{(\xi - \lambda^2 + \lambda) N + \sum_{j=0}^{L-1} \lambda d_j(\lambda d_j - 2(1 - p_j)) - 2\sum_{j=0}^{L-1} (1 - \lambda d_j) \sum_{k=0}^{j-1} (\lambda d_k - (1 - p_k))}{2(L - \lambda N)} \\ & + \frac{1}{N} \sum_{j=1}^{L-1} d_j \sum_{k=0}^{j-1} (\lambda d_k - (1 - p_k)) + \frac{1}{N} \sum_{j=0}^{L-1} d_j \left[\frac{\lambda(d_j - 1)}{2} - (1 - p_j) \right]. \end{aligned} \tag{A.20}$$

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