

Maximal Throughput in a Tandem Multi-Hop Radio Network

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We consider an infinite slotted tandem multi-hop network in which every node is capable of hearing only its two neighbors. The destinations of the packets in the front of the various nodes are independent and independently resampled at each slot whether or not the given node was permitted to transmit. Under the following set of permissible access protocols, the maximal *node utilization* of the network is derived: At the beginning of every slot, each of the nodes may transmit *the first packet in its queue*; the nodes can coordinate their transmissions at no cost and make use of the destination of the first packet at each queue. The transmission scheme that attains the maximum is also specified.

Keywords: Radio Network, Multi-hop, Maximal Throughput, Tandem Network, Access Protocol.

1. Introduction and description of the model

A general multi-hop *packet radio network* consists of a set of nodes N (possibly infinite) and a *hearing range function* $R: N \rightarrow 2^N$. The set $R(n)$ consists of a node n and all the nodes that are able to hear node n whenever it transmits. In particular, $n \in R(n)$ and $n \in R(m)$ implies that $m \in R(n)$ ($n, m \in N$).



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* This research was done while the author was on leave at the IBM Israel Scientific Center.

Nodes attempt to transmit their packets only at slot beginnings and every slot (in time and range) is capable of accommodating a single packet. Every node n has a queue and an arrival process of packets arriving from outside the system to its queue. Each packet is destined for a neighbor m of n ($m \in R(n) - \{n\}$). At the beginning of every slot, each of the nodes may attempt to transmit a packet from its queue (if nonempty). The transmission from n to m is successful if no node k with $m \in R(k)$ tried to transmit during the same slot. If there is such a node k , then k interferes with n . In other words, a node cannot successfully receive a transmission if it is in the range of at least two transmitters (including itself).

If the transmission is successful, we assume that the sender finds this out during the same slot. The transmitted packet is processed at the receiver and then it may join the queue for further transmission or it may leave the system (depending on its ultimate destination). If the transmission is not successful, the sender will retry at some later slot.

Needless to say that if nodes attempt to transmit without any control, there is a high probability of collision. All the techniques that are used to reduce the number of collisions in a fully connected radio network with $R(n) = N$ (i.e., multiple access channel) [7] can be used in a multi-hop network. Roughly speaking, they can be partitioned into three groups: (1) random access protocols as slotted Aloha [6], (2) implicit reservation as TDMA [3], and (3) explicit reservation in which nodes exchange some prior information before transmitting safely. Their properties, as well as performance comparison under different loads, have extensively been studied for a multiple-access channel.

The measure of performance that we are interested in is the maximal *node utilization* of the channel, which is defined as the maximal expected number of stations that transmit successfully per one slot, divided by the number of nodes in the network.

In a general network (not necessarily fully connected), the throughput depends primarily on the range function $R(n)$, $n \in N$, and the access protocol. In a multiple-access channel, the effect of the first source is trivial (at most one node can transmit during every slot). This cannot be said about a multi-hop network.

In this paper we address the maximal *node utilization* problem and focus on the limitations inherent in the range function under a set of access protocols defined below. The constructive method that we use to answer this also gives specific ideas of how to construct good protocols.

Although the node utilization, U , under a given protocol, corresponds to the throughput in the multiple-access channel, one should note that a packet in a multi-hop network might travel more than one hop from source to destination. Therefore, the actual throughput in a multi-hop network, under a given protocol and routing matrix, is the expected number of delivered packets per slot. However, from the mean ergodic theorem (under ergodicity assumptions), the throughput is obtained from U by dividing it by the expected number of hops that a packet travels. Therefore, for a given protocol and routing policy, U does reflect the throughput under that protocol.

For a general multi-hop network, the problem is extremely difficult, so here we focus on an infinite tandem multi-hop network in which each node is capable of hearing only its two neighbors. (A hearing range with distance 1.) For this type of networks, the node utilization problem under specific access protocols has been addressed in [8]. It was pointed out there that under slotted Aloha it is $\frac{4}{27}$ and under TDMA it is $\frac{1}{3}$. These two protocols do not use a vital type of information, the destinations of the packets, currently in front of the buffers. By using this information we derive a maximal node utilization which is significantly better than the numbers above.

The maximal node utilization that we derive in this study is the maximal one under the following set of access protocols:

- At the beginning of every slot, each of the nodes may transmit only *the first packet in its queue* (FCFS).
- The nodes can coordinate their transmission at no cost and make use of the destinations of the first packet in each node's queue.
- The destinations of the packets in the front of the various queues are independent, and independently resampled at each slot whether or not the given node was allowed to transmit (memoryless packet destination).

The model and the protocol assumptions are discussed in Section 4. Note that if one maintains a queue per neighbor or if the node is allowed to search the entire queue before selection (rather than a FCFS

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queue), then it is easy to verify that a time division approach that alternates between the queues has a maximum node utilization of 0.5. While this is true in principle, it does not model a FCFS protocol (which is fair in some sense), nor does it model a protocol one would use in practice under low traffic load, when it is rare that a node desires to send simultaneously to many neighbors. In this situation, a node may have to wait one additional slot if it wants to send towards the opposite direction. Under a low traffic load, this additional waiting slot is significant with comparison to the average waiting time under FCFS. Indeed, under FCFS the average waiting time approaches zero when the load approaches zero, while under the alternating protocol it is at least half a slot (when the probability to transmit to the right is 0.5). Therefore, under low traffic load one would prefer a FCFS protocol.

To conclude, we are interested in a model for FCFS protocols which are used under low traffic load from which we could evaluate how much could the node utilization be pushed up while using these protocols (analogous to random access schemes in a multiple access channel). Also, a FCFS protocol is fair in some sense.

As to the practical motivation of our model, it is clear that understanding the general multi-hop packet radio is in our own interest. However, as in most practical problems, one has to make some simplifying assumptions in order to make the model mathematically tractable and still be able to gain some insight on the original practical problem. With this respect, a tandem network, although not general enough, is a step toward better understanding of the general network.

Nevertheless, a general network can be decomposed into independent tandems by using directed transmitters. Furthermore, tandem networks may also appear in practice. One example is a microwave or a fiber optic link with many repeaters and users. As a second example, one may consider a cellular radio network which provides communication to commuters travelling on a main highway between New York City and Washington D.C. Clearly, the concentrators (supervisor stations) will be arranged in tandem and the communication among themselves can be modeled by our model. A third example could be a network consisting of low power transmitters. Here, the imposed topology is approximately tandem since the transmission range of each node is only sufficient to reach two, one or none of its neighbors. In the last two cases, the network is supplemented with repeaters to gain connectivity.

Regardless of how common tandem networks are, this study points out that one may significantly increase the network utilization by considering protocols that make use of the packet destination, even by confining himself to FCFS protocols. This and other practical issues are discussed in Section 4.

The maximal node utilization under slotted Aloha has also been calculated for networks with regular range function (i.e., in which the nodes are regularly placed on a square grid), homogeneous packets arrivals, and forwarding according to a symmetric routing matrix [6].

The problem of the best TDMA in a general multi-hop network has been addressed in [2], and an approximate analysis for a given TDMA was carried out in [3]. Another related work can be found in [5], where the probability generating function of the buffer occupancies has been derived for two specific tandem networks. The first one is a tandem network in which messages arrive at the top of the network and travel downward till the last node. The other one is a tandem of four nodes in which messages may arrive at every node and then travel downward. Under both types, every node transmits a message if its buffer is not empty.

In Section 2 we formulate a static model, derive the maximal node utilization and develop the main tool for analyzing a dynamic memoryless system. In Section 3 we obtain the maximal node utilization of a stochastic and dynamic network under a memoryless assumption. This maximum is expressed as a function of a parameter which measures the long-run proportion of messages that travels to the right. It is also shown that the optimal node utilization obtains its maximum when the network is symmetric, in which it is 0.404255. We conclude the paper with a brief discussion in Section 4.

2. A static tandem model

In this section we derive the maximal node utilization for a static infinite tandem multi-hop system and develop the main tool to deal with our dynamic system.

In a tandem network of n nodes with a hearing range of distance 1 we have $N = \{1, 2, \dots, n\}$, $R(k) = \{k-1, k, k+1\}$ for $k \neq 1, n$, $R(1) = \{1, 2\}$, and $R(n) = \{n-1, n\}$. At any time slot, every node wants to transmit to the left (a lowered numbered neighbor), right, or not at all. Thus, the state of node k at time t , $d_t(k)$ is, L , R , or \emptyset . Note that the range function is similar for every node except the two edges. To address the asymmetry at the edges, we deal with an infinite tandem $N = \{1, 2, \dots\}$, so that there is only one edge and its contribution is negligible in our symmetric case where all nodes generate traffic at the same rate. The system state at time slot t is $d_t = (d_t(1), d_t(2), \dots)$. A centralized transmission protocol at every time slot t is defined by the following selection function.

A selection (to transmit) S is a function $S: N \rightarrow \{0, 1\}$. For an arbitrary given time slot and $k \in N$, $S(k) = 1$ (respectively 0) means that node k is permitted (respectively, not permitted) to transmit at the time slot.

We say that S is legal at time slot t if:

- (1) for every k , $S(k) = 1$ implies that $d_t(k) \neq \emptyset$,
- (2) for every k_1, k_2 , $S(k_1) = S(k_2) = 1$ implies that nodes k_1 and k_2 do not interfere with each other's transmission.

For every system state $d = (d_1, d_2, \dots)$, $d_k \in \{R, L\} \cup \{\emptyset\}$, and selection S , define the static node utilization as

$$U_S(d) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n S(k) \cdot I\{S(k) \text{ is successful}\}, \tag{1}$$

where $I\{\cdot\}$ is the indicator function of the event $\{\cdot\}$.

Since we are interested in the maximal node utilization, we consider the network under a heavy traffic situation, i.e., in which no queue is ever empty. Clearly, we only have to consider legal selections in which, for every k , $d_k \in \{R, L\}$.

Given a system state d , we would like to find $U^*(d) = \sup_S U_S(d)$, and a selection S^* (if one exists) for which

$$U_{S^*}(d) = U^*(d). \tag{2}$$

The goal of finding S^* is facilitated with the following concepts of intervals and super-intervals. Let $d = (d_1, d_2, \dots)$ be a system state. An interval I starts at node i if $d_i = R$ and $d_{i-1} = L$, and ends at the first node after i , j , where $d_j = L$ and $d_{j+1} = R$. The length of I , $l(I)$, is given as $l(I) = j - i + 1$. Note that an interval of length l consists of k nodes with destination R followed by $l - k$ nodes of destination L , $1 \leq k \leq l - 1$. For example, in the sequence of nodes' states $(\dots, L, (R, R, \dots, R, L, \dots, L), R, \dots)$, the set of nodes within the internal parentheses is an interval. The importance of intervals can be seen from the following lemmas.

2.1. Lemma. Let S be a selection for state d with $S(i) = S(j) = 1$. If $|i - j| \geq 3$, then i and j do not interfere.

Proof. The larger of i and j transmits as far down as $\max\{i, j\} - 1$, and the smaller as far up as $\min\{i, j\} + 1$, which immediately implies the claim of the lemma. \square

Let $\lceil x \rceil$ be the smallest integer greater than or equal to x .

2.2. Lemma. Let S be a legal selection for state d . If I is an interval of length l , then at most $\lceil \frac{1}{3}l \rceil$ nodes i have $S(i) = 1$.

Proof. Suppose, in contradiction, that in a given interval at least $\lceil \frac{1}{3}l \rceil$ nodes can transmit without interference. Then, it is easy to see that at least two of them, i, j ($i \leq j$), be either adjacent or within one node of each other.

If $i = j - 1$, then one of them (or both, if $d(i) = R$ and $d(j) = L$) must receive and transmit at the same time. If $i = j - 2$, then $i + 1$ is destined from at least one of them, but cannot receive due to interferences from the other. \square

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2.3. Lemma. Let d be a system state, $I = (i_1, i_2, \dots, i_l)$ an interval with $l > 2$, and S a legal selection with the following properties:

- (a) If $S(i_1 - 2) = 1$, then $d(i_1 - 2) = L$.
- (b) If $S(i_l + 2) = 1$, then $d(i_l + 2) = R$.

Then there is a legal selection which selects $\lfloor \frac{1}{3}l \rfloor$ nodes of I and agrees with S outside of I .

Proof. Take the selections outside I as before and I as follows: If $l = 3k$ or $3k + 1$, for some $k \geq 1$, then select nodes $1, 3, 7, \dots, 3 \times \lfloor \frac{1}{3}l \rfloor - 2$ of interval I . If $l = 3k + 2$, then select nodes $1, \dots, 3 \times \lfloor \frac{1}{3}l \rfloor - 2, 3 \times \lfloor \frac{1}{3}l \rfloor - 1$, where the intermediate selections are at distance of 3 apart, whenever possible.

From Lemma 2.1 and the definition of interval we obtain a legal selection. \square

Lemmas 2.2 and 2.3 give an approximate characterization of how many nodes can at most transmit in each interval without interfering each other. The lemmas indicate that if we can guarantee independence between intervals (as properties (a) and (b) in Lemma 2.3), then we know that $\lfloor \frac{1}{3}l \rfloor$ nodes can transmit in an interval of length l , and no more than $\lfloor \frac{1}{3}l \rfloor$ can transmit without interference.

From Lemma 2.3 it turns out that in a set of consecutive intervals of length greater than two we may select nodes at every interval in the set, in the same manner as in the proof of Lemma 2.3 and obtain a legal selection on this set. Moreover, this selection is the best selection on this set. Notice that, under this selection, the first and the last node of every interval of length $l \geq 4$, is selected; from intervals with $l = 3$, the first node is selected and, from intervals with $l = 3k$, the last node is never selected.

The problem of interference under the above selection occurs with intervals of length two, since either node, if selected to transmit may interfere with an adjacent interval. For example, consider the set of intervals $\dots, (R, R, L, L), (R, L), (R, R, R, L), \dots$, containing ten nodes and three intervals of lengths 4, 2, 4. It is easy to verify that there is no legal selection which achieves a total of five transmitters as might be hoped for from the above lemmas.

To overcome this situation and for obtaining an independence property among sets of nodes, we introduce the notion of the **super-interval, SI**.

Denote by $I(k)$ an interval whose length is k , by $I(\{3n + j\})$ ($j = 0, 1, 2, n \geq 1$) an interval whose length is a multiple of 3 plus j and by $I^i(2)$ ($i \geq 0$), i consecutive intervals of length two, followed by an interval of length greater than two. Also, (I_1, I_2) means that I_1 and I_2 are two consecutive intervals.

Since pairs (i.e., intervals of length two) cause the difficulties, we will not start a super-interval with a pair, although we may finish them with pairs. The rules of constructing super-intervals are the following:

- (a) The first super-interval starts at the first interval which is not a pair.
- (b) If a super-interval starts with an interval of the form $I(\{3n\})$ or $I(\{3n + 2\})$, then the super-interval consists of the single interval, followed by any number of successive pairs. That is,
 - (b.1) $SI = (I(\{3n\}), I^i(2)), i \geq 0$,
 - (b.2) $SI = (I(\{3n + 2\}), I^i(2)), i \geq 0$,
- (c) If a super-interval starts with $I(\{3n + 1\})$, then it may be one of the following forms:
 - (c.1) $SI = (I(\{3n + 1\}), I^i(2)), i$ is even (including zero).
 - (c.2) $SI = (I(\{3n_1 + 1\}), I^{i_1}(2), I(\{3n_2 + 2\}), I^{i_2}(2), \dots, I(\{3n_k + 2\}), I^{i_k}(2), I(\{3n_{k+1}\}), I^{i_{k+1}}(2))$, i_1, i_2, \dots, i_k are odd and $i_{k+1} \geq 0, k \geq 1$.
 - (c.3) $SI = (I(\{3n_1 + 1\}), I^{i_1}(2), I(\{3n_2 + 2\}), I^{i_2}(2), \dots, I(\{3n_k + 2\}), I^{i_k}(2), I(\{3n_{k+1} + 1\}), I^{i_{k+1}}(2))$, i_1, i_2, \dots, i_k are odd and $i_{k+1} \geq 0, k \geq 1$.
 - (c.4) $SI = (I(\{3n_1 + 1\}), I^{i_1}(2), I(\{3n_2 + 2\}), I^{i_2}(2), \dots, I(\{3n_k + 2\}), I^{i_k}(2), I(\{3n_{k+1} + 2\}), I^{i_{k+1}}(2))$, i_1, i_2, \dots, i_k are odd and i_{k+1} is even, $k \geq 1$.
- (d) The next super-interval starts immediately after the previous one ends.

In words, if an $I(\{3n\})$ or an $I(\{3n + 2\})$ starts a super-interval, we just append the subsequent pairs. If an $I(\{3n + 1\})$ starts a super-interval, then we append all the subsequent pairs. If there is an even number of them, we end the super-interval, and if there is an odd number, then we also append the next non-pair interval and all subsequent pairs. The procedure continues until an $I(\{3n\})$ is found, another $I(\{3n + 1\})$ is found or an $I(\{3n + 2\})$ with an even number of subsequent pairs.

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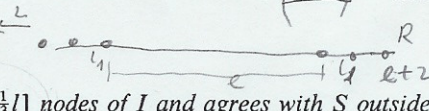
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For every selection S and set of nodes A , let $u_S(A) = \sum_{k \in A} S(k)$. Further, for every interval I and super-interval SI let

$$u^*(I) = \lceil \frac{1}{3}I(I) \rceil \quad \text{and} \quad \bar{u}^*(SI) = \sum_{I \subseteq SI} u^*(I). \tag{3}$$

2.4. Lemma. *Let S be a legal selection for state d . Then, for every super-interval SI ,*

$$u_S(SI) \leq \begin{cases} \bar{u}^*(SI) - 1 & \text{if } SI \text{ is of type (c.3),} \\ \bar{u}^*(SI) & \text{otherwise.} \end{cases} \tag{4}$$

Moreover, there exists a legal selection which obtains the upper bounds in (4) for all the super-intervals in d , and selects the nodes within each super-interval independently from other super-intervals.

Proof. Note that for SI not of the type (c.3), inequality (4) follows directly from Lemma 2.2. Thus, it suffices to prove (4) for SI of type (c.3) and to show how to make a legal selection in an independent fashion which achieves the upper bound.

To see the upper bound for SI of type (c.3), we first observe that, by Lemma 2.2, $u_S(SI) \leq u^*(SI)$, and $u^*(SI)$ may only be achieved by selecting the full $\lceil \frac{1}{3}I(I) \rceil$ in each $I \subseteq SI$. We shall show that this is impossible if S is legal. A super-interval of type (c.3) is of the form

$$SI = (I(3n_1 + 1), I^{i_1}(2), I(3n_2 + 2), I^{i_2}(2), \dots, I(3n_k + 2), I^{i_k}(2), I(3n_{k+1} + 1), I^{i_{k+1}}(2)),$$

where $n_j \geq 1$ for $1 \leq j \leq k + 1$, i_j is odd for $1 \leq j \leq i_k$ and $i_{k+1} \geq 0$.

Now, note that $n_1 + 1$ nodes may transmit in the first interval of SI only if the selected nodes there are 1, 4, 7, ..., $3n_1 + 1$. Since the last transmits, this forces a chain in the i_1 pairs (if one selects the full $\lceil \frac{1}{3}I(I) \rceil$ nodes) that in the 1st, 3rd, 5th, ... pairs the first node transmits, and in the 2nd, 4th, ..., the second node transmits. Any deviation is not a legal selection, or provides less than the full $\lceil \frac{1}{3}I(I) \rceil$. Since i_1 is odd, in the last pair (of the i_1 pairs) the first node transmits, and thus the first node in $I(3n_2 + 2)$ cannot transmit since it will interfere with the last pair. The only remaining way to legally select the full $n_2 + 1$ nodes in $I(3n_2 + 2)$ is to select nodes 2, 5, 8, ..., $3n_2 + 2$. But, this again forces the last node to transmit. As long as there are an odd number of pairs alternating with intervals of the form $I(\{3n + 2\})$, this pattern of forcing continues. For the interval $I(3n_{k+1} + 1)$, the first node is forced not to transmit and a selection of the full n_{k+1} nodes must be obtained from the last $3n_{k+1}$ nodes of the interval. As in the proof of Lemma 2.2, this is impossible. This completes the proof of the upper bound.

To prove the second part of the lemma, we construct an independent legal selection for every super-interval which also obtains the upper bound. This is done by case analysis. (Each case number corresponds to the type of the super-interval.)

(b.1) If i is even, then we select nodes 1, 4, 7, ..., $3n - 2$ from $I(3n)$, the first node from the odd-numbered pairs and the second node from the even-numbered pairs. If i is odd, then we select nodes 1, 4, 7, ..., $3n - 2$ from $I(3n)$, the second node from the odd-numbered pairs and the first node from the even-numbered pairs.

(b.2) If i is even, then we select nodes 1, 4, 7, ..., $3n - 2, 3n + 2$ from $I(3n + 2)$, and from the pairs, as in (b.1) for i even. If i is odd, then we select nodes 1, 4, 7, ..., $3n - 2, 3n + 1$ from $I(3n + 2)$, and pairs as in (b.1) for i odd.

(c.1) Since i is even, we select 1, 4, 7, ..., $3n + 1$ from $I(3n + 1)$, and pairs as in (b.1) for i even.

In (c.2)–(c.4) we begin the selection in the same way: we select nodes 1, 4, ..., $3n_1 + 1$ from $I(3n_1 + 1)$. From the set of the first i_1 pairs we select the first node from the odd-numbered pairs and the second node from the even-numbered pairs. From $I(3n_j + 2)$, $2 \leq j \leq k$, we select 2, 5, 8, ..., $3n_j + 2$. The selection in the last two intervals ends differently for each case.

(c.2) If i_{k+1} is even, then we select nodes 3, 6, ..., $3n_{k+1}$ from $I(3n_{k+1})$, the first node from the odd-numbered pairs and the second node from the even-numbered pairs. If i_{k+1} is odd, then we select

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(c.3) If i_{k+1} is even, then we select nodes $4, 7, \dots, 3n_{k+1} + 1$ from $I(3n_{k+1} + 1)$ (we lose one), and from the pairs as in (c.2) for i_{k+1} even. If i_{k+1} is odd, then the same as in (c.2) for i_{k+1} odd (again we lose one).

(c.4) Since i_{k+1} is even we select $2, 5, \dots, 3n_{k+1} + 2$ from $I(3n_{k+1} + 2)$, the first node from the odd-numbered pairs and the second node from the even-numbered pairs.

Let S^* be the selection which select within each super-interval according the rules above. From the definition, the selection by S^* is done independently between intervals. Also, it is easy to verify that the selection within each super-interval achieves the upper bounds in (4). The selection S^* is legal for every system state d since:

(1) The set of super-intervals is exclusive and exhaustive.

(2) The selection within each super-interval is legal.

(3) The selection within each super-interval starts at the first node of the super-interval and ends at the last node, or at the third to the last (only in case (b.1)). Therefore, there is no interference between super-intervals.

This completes the proof of the lemma. \square

Let S^* be the selection which selects within each super-interval according to rules (b.1), (b.2), (c.1)–(c.4) in the proof of Lemma 2.4. Since S^* achieves the upper bounds within each super-interval we have the following theorem.

2.5. Theorem. For every state d , S^* is the optimal selection. That is,

$$U_{S^*}(d) = \sup_S U_S(d).$$

3. A stochastic memoryless model

In this section we take into consideration the stochastic and dynamic nature of d_t —the system state at time slot t —and adopt a relatively simple memoryless model.

Assume that, for every time slot t and node k ,

$$d_t(k) = \begin{cases} R & \text{with probability } p, \\ L & \text{with probability } 1 - p, \end{cases} \quad (5)$$

where $0 < p < 1$.

For $p = 0$ or 1 , d_t is unidirectional and the optimal selection is clearly every third node. The probability p reflects the long-run proportion of packets that are transmitted to the right-hand node.

We further assume that $d_t(k)$, $t, k = 1, 2, \dots$, are mutually independent. Under heavy load situation (which we consider), the independence among nodes (i.e., the independence of $(d_t(k) | k = 1, 2, \dots)$) is a reasonable assumption. However, the memoryless behavior of $d_t(k)$ at every t is not completely aligned with reality. This assumption is made just for the tractability of the theoretical analysis. In real situations (under heavy load), $d_{t+1}(k)$ is independent of $d_t(k)$ if k transmits during slot t , but does depend if it does not transmit. The situation where $d_{t+1}(k)$ depends on $d_t(k)$ is discussed in Section 4 and investigated in [1].

In the dynamic case, a selection S indicates, for every node k at every slot t , whether node k is permitted or not to transmit at slot t . That is, $S_t(k) = 1$ or 0 , respectively.

A selection S is legal if it is legal for every time slot t . For every legal selection S and t , let

$$U_S(t, p) = \limsup_{n \rightarrow \infty} \frac{1}{n} E_p \left[\sum_{k=1}^n S_t(k) \right],$$

where the expectation is taken with respect to the probability measure of the state d_t introduced by the destination probability p .

part p = p

The long-run node utilization under a legal selection S is given as

$$U_S(p) = \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T U_S(t, p).$$

We are interested in finding

$$U^*(p) = \sup_{S \text{ legal}} U_S(p)$$

and a selection S^* which satisfies

$$U_{S^*}(p) = U^*(p).$$

Again, we consider the system under a heavy traffic situation. When the system is not under heavy traffic (formally under ergodic conditions), every packet which has arrived at the system eventually leaves it. Therefore, $U^*(p)$ is just the average arrival rate per node. Under heavy traffic, $U^*(p) - \epsilon$ (for every $\epsilon > 0$) is the maximal arrival rate per node, under which every packet is transmitted after a finite average delay (provided the selection policy selects each node sufficiently often to handle the traffic which goes through it).

For an arbitrary super-interval SI , let $l(SI) = \sum_{I \subseteq SI} l(I)$ and $u^*(SI)$ be the bounds in (4).

From the renewal property and the independence among super-intervals, $l(SI)$ and $u^*(SI)$ are random variables whose distribution depends only on p .

3.1. Lemma. For every p , $U^*(p) = E_p[u^*(SI)]/E_p[l(SI)]$, and $U^*(p)$ is obtained by selection S^* .

Proof. From the memoryless assumption it suffices to consider only one time step, that is

$$U^*(p) = \sup_S U_S(1, p) = \sup_S \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} E_p \left[\sum_{k=1}^n S_1(k) \right] \right\}.$$

(Hereafter we shall omit the index $t = 1$ from our notation.)

For every realization of the system state $d = (d_1, d_2, \dots)$, let $(SI_k, k \geq 1)$ be a set of consecutive super-intervals. Also, let P_f be the prefix set which is not part of any super-interval. Since $0 < p < 1$,

$$E_p[l(P_f)] < \infty. \tag{6}$$

For every n , let $SI_k, k = 1, 2, \dots, N(n)$, be the set of the consecutive super-intervals which are completely contained in (d_1, d_2, \dots, d_n) and $S_f(n)$ the suffix set which is not part of any super-interval. We have

$$E_p[l(S_f(n))] \leq E_p[l(SI)] < \infty. \tag{7}$$

($E_p[l(SI)]$ is explicitly found below, but we use its finiteness here.)

Let S be any legal selection. Then, from Lemma 2.4 we have, for every n ,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n S(k) &= \frac{1}{n} u_S(P_f) + \frac{1}{n} \sum_{k=1}^{N(n)} u_S(SI_k) + \frac{1}{n} u_S(S_f(n)) \\ &\leq \frac{1}{n} \sum_{k=1}^{N(n)} u^*(SI_k) + \frac{l(P_f) + l(S_f(n))}{n}. \end{aligned} \tag{8}$$

From (6), (7), and (8) it follows that for every $\epsilon > 0$ there exists an n_0 such that

$$\frac{1}{n} E_p \left[\sum_{k=1}^n S(k) \right] \leq \frac{1}{n} E_p \left[\sum_{k=1}^{N(n)} u^*(SI_k) \right] + \epsilon \quad \text{for } n \geq n_0. \tag{9}$$

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From inequality (7), Lemma 2.4 and the mean ergodic theorem (see, e.g., [4, pp. 159–160]), $\lim_{n \rightarrow \infty} (1/n) E_p [\sum_{k=1}^{N(n)} u^*(SI_k)]$ exists and equals

$$E_p [u^*(SI)] / E_p [l(SI)]. \tag{10}$$

Since (9) is valid for every $\epsilon > 0$ and (10) is obtained by S^* , the proof of the lemma is completed. \square

In the sequel we shall derive a closed form for $U^*(p)$ and explore its behavior as a function of p . As we found out in the previous section, $u^*(I)$ and $l(I)$ are closely related to $U^*(SI)$ and $l(SI)$, respectively. Therefore, we start with the corresponding value of (10) for intervals. Let I be an arbitrary interval. Since the intervals are independent, $u^*(I)$ and $l(I)$ are random variables whose distribution depends only on p . Let $r_k(p) = \Pr(l(I) = k)$.

3.2. Lemma. For every $0 < p < 1$ and $k \geq 2$,

$$r_k(p) = \begin{cases} [p(1-p)^k - p^k(1-p)] / [(1-p) - p] & \text{for } p \neq \frac{1}{2}, \\ (k-1)(\frac{1}{2})^k & \text{for } p = \frac{1}{2}. \end{cases}$$

Proof. Let I be an arbitrary interval in system state d , which starts at some node i . Then, it is known that $d_{i-1} = L$ and $d_i = R$. For I to be of length k , one of the $k-1$ following mutually exclusive events has to occur:

$$\{d_{i+j} = R \text{ for } 0 \leq j \leq m; d_{i+l} = L \text{ for } m < l \leq k-1; d_{i+k} = R\}, \quad m = 0, 1, \dots, k-2.$$

Since the destinations are independent we have, from (5), that $r_k(p) = \sum_{j=1}^{k-1} p^j (1-p)^{k-j}$. The lemma is now obtained by straightforward calculation. \square

3.3. Lemma. For every p ,

(a) $E_p [l(I)] = 1 / [p(1-p)].$

(b) $E_p [u^*(I)] = [1 + p(1-p)] / [3p(1-p) + (p(1-p))^2].$

Proof. Let $p \neq \frac{1}{2}$.

(a) By a straightforward calculation,

$$E_p [l(I)] = \sum_{k=2}^{\infty} k \cdot r_k(p) = \frac{1}{1-2p} \left[\frac{1-p}{p} - \frac{p}{1-p} \right] = \frac{1}{p(1-p)}.$$

(b) From (3),

$$E_p [u^*(I)] = \sum_{k=2}^{\infty} [\frac{1}{3}k] r_k(p). \tag{11}$$

For every $0 < \rho < 1$ we have

$$\sum_{k=1}^{\infty} [\frac{1}{3}k] \rho^k = \sum_{j=1}^3 \sum_{i=0}^{\infty} (i+1) \rho^{3i+j} = (\rho + \rho^2 + \rho^3) / (1 - \rho^3)^2. \tag{8}$$

Thus, from Lemma 3.2, equation (11) and standard algebraic manipulation it follows that

$$E_p [u^*(I)] = \frac{1}{1-2p} \left[\frac{1-p}{1-(1-p)^3} - \frac{p}{1-p^3} \right] = \frac{1+p(1-p)}{3p(1-p) + [p(1-p)]^3}. \tag{9}$$

It is easy to verify that the expressions hold for $p = \frac{1}{2}$, too. \square

Let $U_I(p) = E_p[u^*(I)]/E_p[l(I)]$. From Lemma 3.3,

$$U_I(p) = [1 + p(1 - p)] / [3 + (p(1 - p))^2]. \tag{12}$$

In the next lemma we express $U^*(p)$ as a function of $U_I(p)$.

Let \widetilde{SI} be a super-interval of type (c.3), let $\Pr(\widetilde{SI})$ be its probability, and define

$$u^*(\widetilde{SI}) = \sum_{I \subseteq \widetilde{SI}} u^*(I) - 1. \tag{13}$$

For $SI \neq \widetilde{SI}$ define $u^*(SI) = \sum_{I \subseteq SI} u^*(I)$.

3.4. Lemma. For every p , $U^*(p) = U_I(p) - \Pr(\widetilde{SI})/E_p[l(SI)]$.

Proof. For every n , let N_n be the number of intervals till the n th super-interval. From Lemma 3.1 and the independence of the super-intervals,

$$\begin{aligned} U^*(p) &= E_p[u^*(SI)]/E_p[l(SI)] = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n u^*(SI_k) \right] / \left[\frac{1}{n} \sum_{k=1}^n l(SI_k) \right] \\ &= \lim_{N_n \rightarrow \infty} \frac{1}{N_n} \left[\sum_{i=1}^{N_n} u^*(I_i) - \sum_{k=1}^{N_n} \chi(\widetilde{SI}_k) \right] / \left[\frac{1}{N_n} \sum_{i=1}^{N_n} l(I_i) \right], \end{aligned} \tag{14}$$

where

$$\chi(\widetilde{SI}_k) = \begin{cases} 1 & \text{if } SI_k = \widetilde{SI}_k, \\ 0 & \text{otherwise.} \end{cases}$$

The equality in (14) follows from (13) and the fact that $l(SI_k) = \sum_{I \subseteq SI_k} l(I)$.

From (14) and the mean ergodic theorem,

$$\begin{aligned} U^*(p) &= E_p[u^*(I)]/E_p[l(I)] - \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{k=1}^n \chi(\widetilde{SI}_k) \right] / \left[\frac{1}{n} \sum_{k=1}^n l(SI_k) \right] \\ &= U_I(p) - \Pr(\widetilde{SI})/E_p[l(SI)]. \quad \square \end{aligned}$$

To find a closed form for $U^*(p)$ it is left to find $\Pr(\widetilde{SI})$ and $E_p[l(SI)]$, which will be done in the next two lemmas. From the definition in Section 2, a super-interval is an infinite concatenation of dependent 'atoms' A_1, A_2, \dots , which are recursively defined as follows:

$$\begin{aligned} A_1 &= (I(k_1), I^{l_1}(2)), \quad k_1 \geq 3, l_1 \geq 0, \\ A_2 &= \begin{cases} (I(k_2), I^{l_2}(2)), & k_2 \geq 3, l_2 \geq 0 \quad \text{if } k_1 = 3n + 1 \text{ for } n \geq 1, \text{ and } l_1 \text{ is odd,} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

For $i \geq 3$,

$$A_i = \begin{cases} (I(k_i), I^{l_i}(2)), & k_i \geq 3, l_i \geq 0 \quad \text{if } k_1 = 3n_1 \text{ for } n_1 \geq 1, l_1 \text{ is odd, } k_j = 3n_j + 2 \\ & \text{for } n_j \geq 1, \text{ and } l_j \text{ is odd for } j < i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that, in the definition of A_i , the conditioning variables $k_j, l_j, 1 \leq j < i$, are the lengths of the non-pair intervals and their corresponding number of subsequent pairs, respectively, that compose $A_j, j < i$.

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$$(12) \quad \alpha_j(p) = \sum_{k=1}^{\infty} r_{3k+j}(p), \quad j=1, 2, \quad \beta(p) = \sum_{k=0}^{\infty} r_2^{2k+1}(p), \quad q_i = \Pr(A_i \neq \emptyset), \quad i \geq 2,$$

$$q_1 = 1, \quad M(p) = E_p[l(I(\{k\}), I^{(l)}(2))],$$

(13) where $I(\{k\})$ is an interval whose length is any $k \geq 3$ and $I^{(l)}(2)$ is any $l \geq 0$ consecutive pairs followed by an interval of length greater than two. Note that $\alpha_j(p)$ is the probability that an arbitrary interval is of length of the form $3k+j$, $k=1, 2, \dots$, and $\beta(p)$ is the probability of having an odd number of consecutive pair intervals.

A straightforward calculation using Lemma 3.2 and the fact $r_2(p) = p(1-p)$ shows that

$$\alpha_1(p) = [1 - x(p)] / [3 + (x(p))^2], \tag{15a}$$

ma 3.1 and the

$$\alpha_2(p) = [1 - 2x(p) - (x(p))^3] / [3 + (x(p))^2], \tag{15b}$$

$$\beta(p) = x(p) / [1 - (x(p))^2], \tag{15c}$$

(14)

$$M(p) = \sum_{k=3}^{\infty} kr_k(p) / (1 - r_2(p)) + 2 \sum_{k=0}^{\infty} kr_2^k(p)(1 - r_2(p)) = 1 / [x(p)(1 - x(p))] \tag{15d}$$

where $x(p) = p(1-p)$.

Since the probability of an 'atom' of the form $(I(k), I^{(l)}(2))$ ($k = 3n+j$, $j = 1, 2$) for some $n < 1$ and odd l is

$$[\alpha_j(p) / (1 - r_2(p))] \beta(p) (1 - r_2(p)),$$

we have

$$q_i = \alpha_1(p) \beta(p) (\alpha_2(p) \beta(p))^{i-2}, \quad i \geq 2. \tag{16}$$

3.5. Lemma. For every p ,

$$E_p[l(SI)] = M(p) \left[\frac{1 + \alpha_1(p) \beta(p) - \alpha_2(p) \beta(p)}{1 - \alpha_2(p) \beta(p)} \right].$$

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Proof. Since SI is a concatenation of 'atoms' A_1, A_2, \dots , we have, from (16),

$$\begin{aligned} E_p[l(SI)] &= \sum_{i=1}^{\infty} E_p[l(A_i)] = \sum_{i=1}^{\infty} q_i M(p) \\ &= M(p) \left[1 + \alpha_1(p) \beta(p) \sum_{k=0}^{\infty} (\alpha_2(p) \beta(p))^k \right] \\ &= M(p) [1 + \alpha_1(p) \beta(p) / (1 - \alpha_2(p) \beta(p))], \end{aligned}$$

dd,

= $3n_j + 2$

which proves the claim of the lemma. \square

3.6. Lemma. For every p ,

$$\Pr(\widetilde{SI}) = \alpha_1(p)^2 \beta(p) / [(1 - r_2(p))(1 - \alpha_2(p) \beta(p))].$$

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e A_j , $j < i$.

Proof. Let $\widetilde{SI}(i)$ be a super-interval of type (c.3) which consists of $i \geq 2$ 'atoms', i.e., $\widetilde{SI}(i) = (A_1, A_2, \dots, A_i)$. Thus (as obtained for q_i),

$$\begin{aligned} \Pr(\widetilde{SI}) &= \sum_{i=2}^{\infty} \Pr(\widetilde{SI}(i)) = \sum_{i=2}^{\infty} \frac{(\alpha_i(p))^2 \beta^{i-1}(p) (\alpha_2(p))^{i-2}}{1 - r_2(p)} \\ &= \frac{\alpha_1(p)}{1 - r_2(p)} \alpha_1(p) \beta(p) \sum_{k=0}^{\infty} (\alpha_2(p) \beta(p))^k, \end{aligned}$$

which proves the claim of the lemma. \square

From Lemmas 3.4–3.6 and equations (12) and (15) we obtain the following theorem.

3.7. Theorem. For every $0 \leq p \leq 1$,

$$U^*(p) = \frac{3 + 3x(p) - 2(x(p))^2 + (x(p))^3 - (x(p))^4}{9 - (x(p))^4},$$

where $x(p) = p(1 - p)$.

The behavior of $U^*(p)$ as a function of p is given by the next theorem.

3.8. Theorem. (a) $U^*(p)$ is a symmetric and convex function.

(b) $U^*(p) \leq U^*(\frac{1}{2}) = 0.404255$.

Proof. The symmetry is clear from Theorem 3.7. Since $x(p) = p(1 - p)$ is convex, to show convexity of $U^*(p)$ it is sufficient to show that $U^*(p)$ is increasing and convex as a function of $x(p)$ (since, for every function $F(\cdot)$ and $f(\cdot)$,

$$\frac{d^2 F(f(x))}{dx^2} = \frac{d^2 F(f(x))}{d(f(x))^2} \left(\frac{df(x)}{dx} \right)^2 + \frac{d^2 f(x)}{dx^2} \frac{dF(f(x))}{df(x)},$$

whenever the derivatives exist).

Now, it is easy to verify that

$$\frac{dU^*(p)}{dx(p)} < 0 \quad \text{for } 0 \leq x(p) \leq \frac{1}{4}, \quad \text{and} \quad \frac{d^2(U^*(p))}{d(x(p))^2} < 0 \quad \text{for } 0 \leq x(p) \leq \frac{1}{4}.$$

These provide the monotonicity and convexity which completes the proof of part (a).

Part (b) immediately follows from part (a) and some simple computation. \square

4. Discussion

In this section we briefly discuss a few aspects regarding our analysis as well as possible extensions.

4.1. Algorithm to achieve the optimal selection

This paper has concentrated on the calculation of $U^*(p)$ but, clearly, the analysis is constructive in the following way. Given d , one can easily find the optimal selection for d by breaking d down into intervals and super-intervals and selecting as indicated above.

To determine S^* in a distributed manner is somewhat less practical. Since some intervals and super-intervals are arbitrarily long, a node cannot decide whether to transmit until it completes a protocol

i.e., $\widetilde{SI}(i) =$

that travels arbitrarily far. More bandwidth would be used on the protocol than on subsequent message transmission.

Nevertheless, one may get quite close to optimal in a practical sense. The idea would be to have a reservation subslot in which nodes exchange information to determine the length of their super-interval. The exchange would be of bounded distance—any super-interval larger than a particular distance would be approximated by a smaller one. Since the average length of a super-interval is small (only $\frac{16}{3}$ for $p = \frac{1}{2}$, and only about 12 for p as small as 0.1), a reasonable bound on distance may be applied.

Another approach ignores super-intervals and focuses only on intervals, since any node can determine if it starts an interval (using its own and its neighbor's $d(k)$). Since $U_p(p) - U^*(p)$ is small (less than 0.004 for $p = \frac{1}{2}$), even if collisions were suffered at times due to interval-based calculation, the overall performance is not expected to be too badly degraded. Also note that since $E_p(l(I))$ is small (less than 0.004 for $p = \frac{1}{2}$), even if collisions were suffered at times due to interval-based calculation, the overall performance is not expected to be too badly degraded. Also note that since $E_p(l(I))$ is small (4 for $p = \frac{1}{2}$ and only about 11 for p as small as 0.1), in a bounded region one may analyze intervals quite well—and perhaps include some consideration to adjacent pairs.

4.2. Shape of curve

Since the curve of $U^*(p)$ as a function of p symmetric with maximum at $p = \frac{1}{2}$, it is not surprising in the context of our problem that the curve is robust, namely that, for a large range of p 's, $U^*(p) \approx U^*(\frac{1}{2})$. In particular,

$$\begin{aligned} U^*(p) > 0.4 & \quad \text{for } p \in [0.368, 0.632], \\ U^*(\frac{1}{2}) - U^*(p) < 0.01 & \quad \text{for } p \in [0.299, 0.701], \\ U^*(p) \text{ is within 5\% of } U^*(\frac{1}{2}) & \quad \text{for } p \in [0.217, 0.783]. \end{aligned}$$

These facts support the analysis in that the memoryless model only approximates the real system, yet the resulting throughputs do not depend too heavily on p ; hence (we imagine), not too heavily on our various assumptions (equal probabilities at each node, independence between time slots, intermediate routing modeled as part of the arrival process etc.).

4.3. A model with memory

As indicated above, the memoryless model is used only to approximate a realistic system, in which d_{t+1} depends on d_t . It is of interest to see whether this dependency has a negative impact on the node utilization. For this purpose we carried out a simulation of a model with memory where a node which transmits a message pops up a new message to the front of its buffer and the other nodes remain with their previous messages (therefore with their previous destinations). The selection algorithm that we used is the optimal selection in the memoryless model, S^* . The simulation gave a node utilization of 0.4015 for 225 nodes and $p = \frac{1}{2}$, whereas the memoryless analysis gave 0.4043. Moreover, the transmission frequency for each node was approximately equal. These results were reported in [1].

4.4. Other packet radio allocation schemes

If there is a time division allocation of slots to the various nodes (fixed TDMA), it is easy to see that the optimal allocation is to allow every third node to transmit, at a utilization of $\frac{1}{3}$. Thus, the interval analysis provides a substantial improvement, and indicates the value of a nonfixed assignment, but rather one which reacts to the intended direction of each packet. Moreover, a system in which interval analysis is repeated dynamically reacts much better to variabilities in arrival rates than a TDMA scheme.

On the other hand, as mentioned in the Section 1, if one maintains a queue per neighbor or if the node is allowed to search the entire queue before selection (rather than an FCFS queue), then it is easy to verify

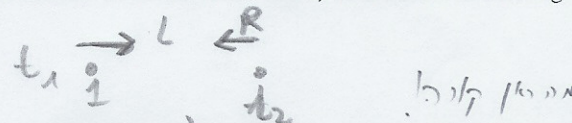


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that a time division approach that alternates between the queues, has a maximum node utilization of 0.5. While this is true in principle, it does not model an FCFS protocol (which is fair in some sense), nor does it model a protocol one would use in practice under low traffic load, when it is rare that a node desires to send simultaneously to many neighbors. In this situation, a node may have to wait one additional slot if it wants to send towards the opposite direction. Under a low traffic load, this additional waiting slot is significant with comparison to the average waiting time under FCFS. Indeed, under FCFS, the average waiting time approaches zero when the load approaches zero, while under the alternating protocol it is at least half a slot (when the probability to transmit to the right is 0.5). Therefore, under low traffic load one would prefer an FCFS protocol.

We are interested in a model for FCFS protocols which are used under low traffic load from which we could evaluate how much the node utilization could be pushed up while using these protocols (analogous to random access schemes in a multiple access channel.) Also, an FCFS protocol is fair in some sense.

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Throughput Network Service

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1. Introduction

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