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Maximal Selection in Tandem Networks with Symmetric Hearing Range[1]

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Abstract. We consider an infinite tandem network in which every node is capable of hearing its neighbors up to a given distance n. At any moment of time every node may contain in the top of its queue a message destined to one of its neighbors. This network can be used as a model for a microwave or optic link with many users. For small and large n we investigate the maximal selection of nodes in the network, for which their transmissions are collision-free. For a large hearing range we show that the upper bound on the maximal selection, which is found herein, is asymptotically achievable. For small hearing ranges we show that a greedy selection is better but not asymptotically optimal. We also specify a sequence of upper bounds which converge to the maximal throughput.

Key Words. Tandem networks, Maximal utilization, Multihop communication networks, Collision-free algorithm.

1. Introduction and Description of the Model. A general multihop *packet radio network* consists of a set of nodes N (possibly infinite) that access a shared channel, where each node has its own hearing range. That is, for every node n, $n \in N$, there is a set of nodes $R(n)$ that consists of all nodes that are able to receive node n whenever it transmits. The set $R(n)$ is referred to as the neighbors of node n.

We assume that the channel is slotted, nodes attempt to transmit their messages only at slot beginnings and every slot (in time and range) is capable of accommodating a single packet. Every node has its own queue and arrival process of packets arriving from outside the system. Each packet at the queue of node n is destined for a single neighbor of n. At the beginning of every slot each of the nodes may attempt to transmit a packet from its queue (if nonempty). The *transmission from n to m is successful* if no other neighbor of m, nor node m, tried to transmit during the same slot. If there is such a node k, then k *interferes* with n. In other words, a node cannot successfully receive a transmission if it is in the range of at least two active transmitters (including itself).

If the transmission is successful, we assume that the sender finds this out during the same slot. The successfully received packet is processed at the receiver and then it may join the queue for further transmission or it may leave the system (depending on its ultimate destination). If the transmission is not successful, the sender will retry at some later slot.

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Needless to say, that if nodes attempt to transmit without any control, there is a high probability of collision. All the techniques that are used to reduce the number of collisions in a single-hop network ($R(n) = N$, for every n) [To] can be used in a multihop network. Roughly speaking, they can be partitioned into three groups:

(1) Random-access protocols as slotted Aloha [SiKl].
(2) Implicit reservation as TDMA [NeKl].
(3) Explicit reservation in which nodes exchange some prior information before transmitting safely.

Their properties, as well as performance comparison under different loads, have been extensively studied for a multiple-access channel.

The measure of performance that we are interested in is the maximal *node utilization*, which is defined as the maximal expected number of nodes that transmit successfully per one slot, divided by the number of nodes in the network.

In a general multihop radio network, the throughput depends primarily on the hearing ranges of the nodes (the sets $R(n)$, $n \in N$) and the access protocol that is being used. In a single-hop network, the effect of the first source is trivial (at most one node can transmit during every slot). This cannot be said about a multihop network. In this paper we address the maximal *node utilization* problem and focus on the limitations inherent in the range function under a set of access protocols defined below. The constructive method that we use to answer this also gives specific ideas of how to construct distributed protocols.

Although the node utilization, U, under a given protocol, corresponds to the throughput in the multiaccess channel, we should note that a packet in a multihop network might travel more than one hop from source to destination. Therefore, the actual throughput in a multihop network under a given protocol and routing matrix, is the expected number of packets that are delivered per slot to their final destination. However, from the mean ergodic theorem (under ergodicity assumptions), the throughput is obtained from U by dividing it by the expected number of hops that a packet travels. Therefore, for a given protocol and routing policy, U does reflect the throughput under that protocol.

For a general network, the problem of finding a transmission algorithm which maximizes the *node utilization* is known to be NP-complete [EGMT], so we focus here on an infinite tandem network with a symmetric hearing range of distance n. That is, all nodes are arranged in tandem and every node is able to hear its neighbors of each side up to distance n. In such a network we have $N = \{1, 2, \ldots\}$, $R(k) = \{k-n, k-n+1, \ldots, k-1, k, k+1, \ldots, k+n-1, k+n\}$ for $k > n$, and $R(k) = \{1, 2, \ldots, k, k+1, \ldots, k+n\}$ for $k \leq n$. At any time slot, every node wants to transmit to one of its neighbors from the left or the right, or not at all.

For this type of network with $n = 1$, the node utilization problem under specific access protocols have been addressed in [Ye]. It was pointed out there that under slotted Aloha it is $\frac{4}{27}$ and under TDMA it is $\frac{1}{3}$. (For the definition of the above policies, the reader is referred to [SiKl].) These two protocols do not use vital type of information, the destination of the packets. By using this information it was shown in [JaRo2] that, for $n = 1$, the maximal node utilization is

0.404255 .... In addition, a protocol was given to select those nodes that must transmit in order to achieve the maximum utilization.

In this paper, which is an extension of [JaRo2], we address the maximal *node utilization* problem for $n > 1$ under the following set of access protocols:

- At the beginning of every slot, each of the nodes may transmit only *the first packet in its queue* (FCFS).
- The nodes can coordinate their transmissions at no cost, and make use of the destinations of the first packet in each node's queue.
- The destinations of the packets in the front of the various queues are independently resampled at each slot, whether or not the given node was allowed to transmit. (Memoryless packet destinations.)

The model and the protocol assumptions are discussed in Section 5. As to the practical motivation of the infinite tandem topology, it appears to be a good model for a microwave or fiber optic link with many users. Also, a general network can be decomposed into independent tandems by using directed transmitters. Our ultimate interest is to understand the general multihop packet network. However, as in most practical problems, we have to make some simplifying assumptions in order to make the model mathematical tractable and still be able to gain some insight on the original practical problem. As in other complex problems, an effective approach is decomposition. A general network may be decomposed into tandems, and the behavior of each tandem can be approximated by viewing it as it operates in isolation. Furthermore, a tandem is a building block in networks which capture transmission interference.

The maximal node utilization under slotted Aloha has been calculated for networks in which the nodes are regularly placed on a square grid, homogeneous packets arrivals, and forwarding according to a symmetric routing matrix [SiKl].

The problem of the best TDMA in a general multihop network have been addressed in [Ne], and an approximate analysis for a given TDMA was carried out in [NeKl]. Another related work is [Si], in which the probability generating function of the buffer occupancies have been derived for two specific tandem networks. The first one is a tandem network in which packets arrive at the top of the network and travel downward till the last node. The other one is a tandem of four nodes in which packets may arrive at every node and then travel downward. Under both types, every node transmits a packet if its buffer is not empty.

In Section 2 we derive a collision-free selection algorithm which is asymptotically optimal. In Section 3 we suggest a greedy algorithm and analyze its performance. In Section 4 we give a sequence of upper bounds to the maximal *node utilization* from which it turned out that the greedy algorithm is quite close to the upper bound for a small hearing range. In Section 5 we discuss the model, the protocols assumptions, and their distributed implementation.

The reader should note that the algorithms and the analysis which are presented here, are solely developed to show that the upper bound is achievable, and we do not suggest implementing them as they are. They are all based on central information and therefore cannot be implemented distributively. We discuss the distributed implementation in Section 5.

2. An Asymptotically Optimal Algorithm. In the first subsection we formalize the model and in the second we derive the asymptotically optimal algorithm.

2.1. Formalization of the Model. In an infinite tandem network with a hearing range n we have $N=\{1, 2, \ldots\}$,

$$R(k)=\{k-n,\ k-n+1, \ldots, k-1,\ k,\ k+1, \ldots, k+n-1,\ k+n\}$$

for $k>n$, and $R(k)=\{1, 2, \ldots, k, k+1, \ldots, k+n\}$ for $k\leq n$. For an arbitrary given node, let R_i, L_i, $1\leq i\leq n$, denote the transmission destinations relative to that node. Here, R_i and L_i are the nodes at distances i to its right and to its left, respectively. That is, for node $k>n$, R_i and L_i stand for nodes $k+i$ and $k-i$, respectively. At any time slot, the destination of the front packet (if it exists) at any given node is an element in the set $\{R_i, L_i \mid 1\leq i\leq n\}$ and is referred to as the *state* of that node at time t. If the buffer at that node is empty, we say that the node is at state φ.

Let $d_t(k)$ be the *state* of node k at time slot t, and $\mathbf{d}_t=(d_t(1), d_t(2), d_t(3), \ldots)$ be the *system state* at time slot t.

A *selection* (to transmit) S is a function $S\colon N\to\{0, 1\}$. For every node k, $S(k)=1$ (0) means that node k is permitted (not permitted) to transmit. We say that a selection S is *legal* at time slot t if:

(1) For every k, $S(k)=1$ implies that $d_t(k)\neq\varphi$.
(2) For k_1, k_2 with $S(k_1)=S(k_2)=1$, implies that nodes k_1 and k_2 do not interfere with each other's transmission.

A legal selection S is *maximal* if for any legal selection $\hat{S}$, $\{k \mid S(k)=1\}\subseteq\{k \mid \hat{S}(k)=1\}$ implies $S=\hat{S}$. That is, in a maximal selection we cannot select additional nodes without resulting in an interference.

For the sake of clarity, consider the following example with $n=2$, and observe the states and the selections of nodes $101, 102, \ldots, 110$. A feasible state is

$$\begin{array}{lccccccccccl} \mathbf{d}=(\cdots & L_1 & R_1 & R_2 & L_2 & L_1 & L_1 & R_1 & R_2 & L_1 & R_1 & \cdots) \\ \quad\ \cdots & 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 & 109 & 110 & \cdots \end{array}$$

The state $\mathbf{d}$ represents a situation where the front packet at node 101 is destined to the next node to its left (node 100). The front packet at node 102 is destined to the next node to its right (node 103), and so forth until node 110, whose front packet is destined to the next node to its right (node 111).

In this example, a selection S for which $S(101)=S(102)=1$ is illegal, since the receiving node 103 can hear the transmissions of nodes 101 and 102. A selection S for which $S(102)=S(103)=1$ is also illegal, since the receiving node 103 is transmitting at the same time. A selection S that selects only nodes 101 and 109 from nodes $101, 102, \ldots, 110$, could be legal but not maximal, since we may select node 105 without resulting in any interference.

Since we are interested in the maximal node utilization, we consider a heavy-traffic situation in which at every time slot every node has always something to transmit. We further assume that $d_t(k)$, $t, k = 1, 2, \ldots$, are mutually independent, and for every time slot t, node k $(k > n)$ and $i = 1, 2, \ldots, n$,

$$d_t(k) = \begin{cases} R_i & \text{with probability } p/n, \\ L_i & \text{with probability } (1-p)/n, \end{cases}$$

where $0 \le p \le 1$ reflects the long-run probability that a node transmits to the right.

For every system parameters (n, p), system state $\mathbf{d} = (d(1), d(2), \ldots)$, and *selection* S, define the *node utilization* of state $\mathbf{d}$ as

$$(2.1)\quad U_S(n, p, \mathbf{d}) = \limsup_{K \to \infty} \frac{1}{K} \sum_{k=n+1}^{K} S(k) \cdot I\{\text{transmission of node } k \text{ is successful}\},$$

where $I\{\cdot\}$ is the indicator function of the event $\{\cdot\}$. Since we have an infinite number of nodes, the definition in (2.1) is the same, if we had taken the summation from one. Also, when the limit exists, $U_S(n, p, \mathbf{d})$ is the proportion of the successful transmissions.

A selection algorithm S indicates, for every node k at every time slot t, whether or not the node is permitted to transmit at slot t. That is, $S = \{S_t\}$, and S_t is a selection for every slot t, and can be chosen as a function of the system state at time t, $\mathbf{d}_t$. A selection algorithm S is legal if, for every time slot t, S_t is legal.

Under our model assumptions, for every selection algorithm the system states $\mathbf{d}_t$, $t = 1, 2, 3, \ldots$, are independent random variables. This implies that in order to maximize the long-run average node utilization, we may restrict ourselves to stationary selection algorithms. That is, $S_t = S$ for every t. Furthermore, the long-run average node utilization of a system with parameters (n, p), under a stationary selection algorithm S, is

(2.2)

$$U_S(n, p) = \limsup_{K \to \infty} \frac{1}{K} E_p\left[\sum_{k=n+1}^{K} S(k) \cdot I\{\text{transmission of node } k \text{ is successful}\}\right].$$

For a given S, the expectation is taken with respect to the distribution of the system state, and depends on the parameter p.

We are interested in finding

$$U^*(n, p) = \sup_S U_S(n, p),$$

and a selection algorithm S^* (if one exists) which satisfies

$$(2.3)\quad U_{S^*}(n, p) = U^*(n, p).$$

Clearly, we have to consider only legal selections.

2.2. Asymptotically Optimal Algorithms. In this subsection we first derive an upper bound to the maximal node utilization, and then we show that it is asymptotically achievable by a legal selection. Let (n, p) be any given system parameters. In the next theorem we show that on the average, the number of nodes between two selected transmitters is at least n.

THEOREM 2.1. *For any system state* $\mathbf{d}$ *and legal selection S,* $U_S(n, p, \mathbf{d}) \le 1/(n+1)$.

PROOF. We consider each node k_1 that is selected to transmit under S ($S(k_1)=1$), and show that if the next transmitter to the right, k_2, satisfies $k_2 - k_1 < n+1$, then the following transmitter to the right, k_3, satisfies $k_3 - k_1 \ge 2n+2$. This allows us to partition all transmitters into singletons that are followed by n nontransmitters, and pairs of transmitters who between them have $2n$ nontransmitters following them. This clearly implies the theorem.

Suppose that $k_2 - k_1 < n+1$. Then $d(k_2) \ne L_j$ for any j, since otherwise k_1 interferes with k_2's transmission. Moreover, $d(k_2)$ must be R_j with $j \ge n-(k_2-k_1-1)$, which implies that $k_3 \ge k_2 + j + n + 1$ in order not to interfere with k_2. Thus $k_3 - k_1 \ge k_2 + j + n + 1 - k_1 \ge 2n+2$. □

For every tandem network with parameters (n, p), we show that the upper bound is asymptotically achievable. This is done by describing a method to arrive at a legal selection algorithm S^n, that is asymptotically optimal in the sense that

$$U_{S^n}(n, p) = \frac{1}{n + o(n)^4}.$$

That is to say, that on the average the number of nodes between two selected transmitters is asymptotic to n.

The idea of the asymptotically optimal algorithm is as follows. For every hearing range n, and arbitrary k, $1 \le k \le n$, we define a selection algorithm S^n that scans the tandem from left to right and alternates between selecting nodes that transmit to the right, and nodes that transmit to the left. The algorithm selects only nodes that transmit to destinations not larger than k. To obtain a legal selection, each subsequent transmitter is selected as follows: if the next selected node has to transmit to its left, this transmission should not interfere with one originating at the preceding selection (i.e., from a node to its left), assuming this latter transmission is destined to a node at distance k to that node's right. Similarly, if the next selected node has to transmit to its right, this transmission should not interfere with one originating at the preceding selection, assuming this latter transmission is destined to a node at distance 1 to that node's left. Formally, selection S^n is defined as follows.

DEFINITION 2.1. The selection algorithm S^n scans the tandem from left to right and alternates between selecting nodes l with $d(l) \in R^k = \{R_1, R_2, \ldots, R_k\}$ and $d(l) \in L^k = \{L_1, L_2, \ldots, L_k\}$ for some k. The first selected node, l_1, is the first node with $d(l_1) \in R^k$. The second node that is chosen is the first node $l_2 > l_1 + n + k$ with $d(l_2) \in L^k$. The third node is the first node $l_3 \ge l_2 + n$ with $d(l_3) \in R^k$. Recursively, we define the selected nodes as follows. For i even, the ith selected node is the first node $l_i > l_{i-1} + n + k$ with $d(l_i) \in L^k$. For i odd, the ith selected node is the first node $l_i \ge l_{i-1} + n$ with $d(l_i) \in R^k$.

[4] $f(n)$ is $o(n)$ if $\lim_{n\to\infty}(f(n)/n) = 0$.

For clarification, consider an example with $n=3$, $k=2$, and a state

(2.4) $\mathbf{d}=(L_1\ L_2\ R_2\ R_1\ L_1\ R_3\ L_2\ R_3\ L_3\ L_3\ L_1\ L_3\ R_3\ R_3\ L_1\ R_1\ \cdots),$
$\quad\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \cdots$

where the nodes are numbered from left to right by $1, 2, 3, \ldots$. The nodes that are selected by S^3 are nodes 3, 11, and 16. Note that node number 9 is not selected, although it does not interfere with other nodes' transmissions. The reason is that we use $k=2$, and the state of node number 9 is L_3. For $k=3$, this node would have been selected by S^3. Similarly, node 14 is not selected since its state is R_3.

The idea behind the algorithm is to select nodes at distances bounded by $n+k+\tau$, where τ is a random distance that is geometrically distributed. Then, by a proper choice of k we can have interselection distances whose expectations is of order of n. This intuitive result is proven in the next theorem and its associated corollaries. First we prove, however, that S^n is legal.

LEMMA 2.1. *The selection algorithm S^n is legal.*

PROOF. For i even (where transmissions are to the left-hand side), no transmission interferes with l_i, since the transmission is to a node in $\{l_i-k, \ldots, l_i-1\}$. Indeed, for $j<i$, all transmitters satisfy $l_j+n+k<l_i$ and thus $l_j+n<l_i-k$ (i.e., their transmissions do not reach l_i-k). For $j>i$, all transmitters satisfy $l_j\ge l_i+n$, and thus $l_j-n\ge l_i$ (i.e., their transmissions do not reach l_i-1).

For i odd (where transmissions are to the right-hand side), no transmission interferes with l_i since the transmission is to a node in $\{l_i+1, \ldots, l_i+k\}$. Indeed, for $j<i$, all transmitters satisfy $l_j+n\le l_i$, and therefore their transmissions do not reach l_i+1. For $j>i$, all transmitters satisfy $l_j>l_i+n+k$, and thus $l_j-n>l_i+k$ and their transmissions do not reach l_i+k. □

In order to find $U_{S^n}(n,p)$, we use the mean ergodic theorem [Ro, pp. 159–160].

THEOREM 2.2. *For every $0<p<1$,*

$$U_{S^n}(n,p)=1\Big/\left(n+\frac{k-1}{2}+\frac{n}{2k}\left(\frac{1}{p}+\frac{1}{1-p}\right)\right).$$

PROOF. From (2.2) we have

$$U_{S^n}(n,p)=\limsup_{K\to\infty} E\left[\frac{1}{K}(\text{the number of transmitters until node } K)\right]. \tag{2.5}$$

First note that the odd selections are renewal points in the selection process of S^n. That is, the number of nodes, Z, between any two consecutive odd-selected nodes are independent identically distributed random variables. From the definition of S^n we have

$$E(Z)=(n+k)+E(X)+(n-1)+E(Y),$$

where X and Y are geometric random variables with expectations $n/k(1-p)$ and n/kp, respectively. Note that $(n+k)$ is due to an even selection and $(n-1)$ is due to an odd selection. Hence, $E(Z)=n+k+n/(k(1-p))+n-1+n/kp$. From the mean ergodic theorem (see, e.g., pp. 159–160 of [Ro]), the limit in (2.5) exists and equals

$$\frac{2}{E(Z)}=1\bigg/\left(n+\frac{k-1}{2}+\frac{n}{2k}\left(\frac{1}{p}+\frac{1}{1-p}\right)\right). \qquad \square$$

From the theorem above, it is easy to verify that by choosing k as in the following corollary, we obtain the desired asymptotic behavior.

Corollary 2.1. *If* $k=\lceil(n((1/p)+(1/(1-p))))^{1/2}\rceil$, *then*

$$U_{s^n}(n,p)\geq 1\bigg/\left(n+\left(n\left(\frac{1}{p}+\frac{1}{1-p}\right)\right)^{1/2}\right),$$

where $\lceil x\rceil$ *is the smallest integer larger than or equal to* x.

Corollary 2.2. *If* $p=\frac{1}{2}$, *then* $U_{S^n}(n,\frac{1}{2})\geq 1/(n+2n^{1/2})$.

Corollary 2.3.

(i) *If p is constant, then* $U_{S^n}(n,p)=1/(n+O(n^{1/2}))=1/(n+o(n))$.
(ii) *If p is variable (depending on n) and* $1/p(n)=o(n)$, *or* $1/(1-p(n))=o(n)$, *then* $U_{S^n}(n,p)=1/(n+o(n))$.

While S^n is asymptotically optimal, it is not *maximal*. This is because we may be "searching" for L^k (or R^k) for a long time. For example, if we search for more than $4n+3$ nodes after a particular transmitter, we can clearly add the $(2n+1)$st node to S^n and still be legal. The *maximal* property of a selection algorithm is only important for small n and therefore will be used to improve the lower bound when n is small.

An improved algorithm $\hat{S}^n$ starts by choosing precisely those nodes that S^n chooses. Then between two transmitters it determines if transmitters may be added without interfering with existing transmitters. If they may be added, they are added in a greedy fashion (i.e., one sequentially examines nodes 1, 2, 3, . . . and adds whichever do not interfere—see Section 3).

Clearly, $\hat{S}^n$ is *maximal* and its *node utilization* is at least $U_{s^n}(n,p)$, hence asymptotically optimal. We do not calculate $U_{\hat{S}^n}(n,p)$ explicitly here, but point out that the addition is very valuable for small values of n. For example, for $p=\frac{1}{2}$ and $n=1$, $U_{S^n}(n,p)=\frac{1}{3}$ and $U_{\hat{S}^n}(n,p)=0.3809$.

Another problem with S^n is the severe deficiency that it only chooses transmitters with states in L^k or R^k. In a nonrenewal situation, after such nodes transmit there would be fewer nodes in the next time interval in L^k and R^k, and thus utilization would decrease. The next algorithm, $\tilde{S}^n$, describes how to fix that by giving equal probability to each state of transmitter and hence proves that the upper bound is also asymptotically achievable by a "fair" selection algorithm.

The idea of the "fair" algorithm is as follows. We first partition the set of relative destinations $\{R_1, R_2, \ldots, R_n\}$ into disjoint sets, each containing k destinations, except for possibly one. The same is done for the set of distances $\{L_1, L_2, \ldots, L_n\}$. Formally, the partition is defined as follows. For every n, $1 \le k \le n$ and $1 \le j < \lceil n/k \rceil$, let

$$R^{j\cdot k} = \{R_{(j-1)k+1}, \ldots, R_{jk}\}, \qquad L^{j\cdot k} = \{L_{(j-1)k+1}, \ldots, L_{jk}\}.$$

For $j = \lceil n/k \rceil$, let

$$R^{j\cdot k} = \{R_{(j-1)k+1}, \ldots, R_n\}, \qquad L^{j\cdot k} = \{L_{(j-1)k+1}, \ldots, L_n\}.$$

For convenience of notation we use a version of the modulo notation. For every positive integer t, let $[t] = 1 + (t-1) \bmod \lceil n/k \rceil$. Note that $[t]$ assumes only the values $1, 2, \ldots, \lceil n/k \rceil$. That is, $[i \cdot \lceil n/k \rceil + m] = m$.

Now, the selections of $\tilde{S}^n$ are similar to those of S^n, with the difference that $\tilde{S}^n$ is not restricted to destinations in the sets $\{R_1, R_2, \ldots, R_k\}$, $\{L_1, L_2, \ldots, L_k\}$. Selection $\tilde{S}^n$ alternates among the sets $R^{j\cdot k}$, $L^{j\cdot k}$, $1 \le j \le \lceil n/k \rceil$, in a round-robin manner. Hence, it does not discriminate nodes with respect to their destinations. The selection criterion within a set $R^{j\cdot k}$, $L^{j\cdot k}$ is similar to the criterion that is used by S^n. Moreover, whenever the algorithm selects a node with a destination in the set $R^{1\cdot k}$, the algorithm assumes that the preceding selection is transmitting to distance L_1. This makes the selections of every $2\lceil n/k \rceil$ consecutive nodes independent. The formal definition of the algorithm is given below.

Definition 2.2. The algorithm $\tilde{S}^n$ starts as S^n by selecting the first node l_1 with $d(l_1) \in R^{1\cdot k}$ and then the first node $l_2 > l_1 + n + k$ with $d(l_2) \in L^{1\cdot k}$. The rest of the selections are defined recursively as follows. For i even and not equal to $2j\lceil n/k \rceil + 2$ for some $j \ge 1$, the ith selected node is the first node $l_i > l_{i-1} + n + [i/2]k$ with $d(l_i) \in L^{[i/2]\cdot k}$. For i even and equal to $2j\lceil n/k \rceil + 2$ for some $j \ge 1$, the ith selected node is the first node $l_i > l_{i-1} + 2n$ with $d(l_i) \in L^{[i/2]\cdot k}$. For i odd and not equal to $2j\lceil n/k \rceil + 1$ for some $j \ge 1$, the ith selected node is the first node $l_i \ge l_{i-1} + n - ((i-3)/2 \bmod \lceil n/k \rceil)k$ with $d(l_i) \in R^{[(i+1)/2]k}$. For i odd and equal to $2j\lceil n/k \rceil + 1$ for some $j \ge 1$, the ith selected node is the first node $l_i \ge l_{i-1} + n$ with $d(l_i) \in R^{1\cdot k}$.

For the sake of clarity, consider an example with $n = 3$ and $k = 2$. Here, $R^{1\cdot k} = \{R_1, R_2\}$, $R^{2\cdot k} = \{R_3\}$, $L^{1\cdot k} = \{L_1, L_2\}$, and $L^{2\cdot k} = \{L_3\}$. For the system state in (2.4), algorithm $\tilde{S}^n$ selects nodes 3, 11, and 14.

The asymptotic properties of $\tilde{S}^n$, as well as the proofs, are similar to those of S^n. These are given in the next theorem and its associated corollary. The proof that this algorithm is legal goes along the same lines as the proof of Lemma 2.1, and therefore it is left to the reader.

Lemma 2.2. *The selection algorithm $\tilde{S}^n$ is legal.*

THEOREM 2.3. *For every* $0<p<1$,

$$U_{\tilde{S}^n}(n,p)\geq 1\Big/\left[n+\frac{n}{2k}\left(\frac{1}{p}+\frac{1}{1-p}\right)+\frac{k}{2}\left(1+\frac{n}{n-k}\right)-\frac{1}{2}-\frac{k^2}{2n}\right].$$

PROOF. From (2.2) we have

$$U_{\tilde{S}^n}(n,p)=\limsup_{K\to\infty} E\left[\frac{1}{K}\,(\text{the number of transmitters until node } K)\right]. \tag{2.6}$$

Note that every $2j\lceil n/k\rceil+1$ selected node ($j=0,1,2,\ldots$) is a renewal point in the selection process of $\tilde{S}^n$. That is, the next $2\lceil n/k\rceil$ selections are independent of the previous ones, and their states and interselection distances are distributed as the previous $2\lceil n/k\rceil$ selections. Let Z be the number of nodes between two consecutive renewal points (including the left one and not including the right one). From the definition of $\tilde{S}^n$ we have

$$E(Z)=(n-1)+E(Y_1)+\sum_{i=2}^{\lceil n/k\rceil}[(n-1-(i-2)k)+E(Y_i)]$$
$$+\sum_{i=1}^{\lceil n/k\rceil-1}[(n+ik)+E(X_i)]+2n+E(X_{\lceil n/k\rceil}).$$

The random variables X_i and Y_i are geometrically distributed, and for $i<\lceil n/k\rceil$ their expectations are $n/k(1-p)$ and n/kp, respectively. For $i=\lceil n/k\rceil$, their expectations are $n/k(1-p)$ and n/kp, respectively.

Therefore,

$$E(Z)\leq\lceil n/k\rceil(n-1)+\lceil n/k\rceil n+(\lceil n/k\rceil-1)k+n+\lceil n/k\rceil(n/kp+n/k(1-p)). \tag{2.7}$$

Since between every renewal points there are $2\lceil n/k\rceil$ transmitters, it follows from the mean ergodic theorem that the limit in (2.6) exists and equals $2\lceil n/k\rceil/E(Z)$. The theorem now follows from (2.7). □

COROLLARY 2.4. *If* $k=\lceil((n/2)(1/p+1/(1-p)))^{1/2}\rceil$, *then, for every* $\varepsilon>0$ *and* n *sufficiently large,*

$$U_{\tilde{S}^n}(n,p)\geq 1/(n+(2+\varepsilon)\cdot c\cdot n^{1/2}),$$

where $c=(\frac{1}{2}(1/p+1/(1-p)))^{1/2}$.

We see that $\tilde{S}^n$ is not much worse than S^n. Moreover, we show in the next theorem that it equalizes the transmission probabilities of each node.

THEOREM 2.4. *If* k *divides* n, *then the expected number of transmitters with* L_i *or* R_i, *which are selected by* $\tilde{S}^n$, *is* $1/2n$.

PROOF. By construction, each group $R^{1\cdot k}, L^{1\cdot k}, \ldots, R^n, L^n$ has an equal number of transmitters. Within each group, each group member has an equal probability of choice by the independence of all values of $d(i)$. □

We note that in a nonrenewal system the selection $\tilde{S}^n$ has a different bias, since $L^{1\cdot k}$ and $R^{1\cdot k}$ always come from the first few nodes, $L^{2\cdot k}$ and $R^{2\cdot k}$ from the next few, etc. This can easily be handled by rotating the first nodes' group. Also, $\tilde{S}^n$ can easily be modified (as was S^n) to make it maximal.

3. A Greedy Algorithm. In Section 2 it was shown that the selection algorithms S^n, $\hat{S}^n$, and $\tilde{S}^n$ are asymptotically optimal. However, for a small hearing range they are far from being satisfactory. For example, if $n = 1$ and $p = \frac{1}{2}$ the maximum *node utilization* is 0.404255 (see [JaRo2]) but Theorem 2.1 provides an upper bound of 0.5 and S^1 yields a *node utilization* of $\frac{1}{3}$. Even $\hat{S}^n$ only achieves 0.3809. For $n = 2$, the upper bound is $\frac{1}{3}$ and S^2 yields 0.2222. These differences between the upper bound and the *node utilization* of S^n for small n's suggests that we should be able to decrease the upper bound and provide a better algorithm for small n's. The improvement for small n's is achieved by the following greedy selection algorithm, S_g.

3.1. The Utilization for Small n's. In this subsection we define the greedy selection algorithm and evaluates its *node utilization*. The idea of the greedy algorithm is apparent from its definition below.

DEFINITION 3.1. The algorithm S_g scans the tandem from left to right. The first selected node, l_1, is node 1. For $i > 1$, the ith selected node, l_i, is the first node with $l_i > l_{i-1}$ which does not interfere with the previous selections.

Clearly, the algorithm is legal and at every step it selects a node to minimize the current interselection distance. In this selection process, no consideration is given to the effect of the state of the current selected node on subsequent selections.

REMARK 3.1. Note that if l_i and l_{i-1} do not interfere with each other's transmissions, then l_i and l_j similarly do not interfere for every $j < i$.

In the sequel we shall evaluate the *node utilization* of S_g for any given hearing range n. Note that the node states are random and the selection process is performed from left to right. Furthermore, every additional selection depends only on the state of the current selected node. Therefore, the states of the selected nodes and their associated interselection distances is a semi-Markov process. A semi-Markov process is defined by a state space D, a transition probability matrix P, and a transition time between two consecutive steps T.

The state space D is taken to be the set of all possible transmission distances. That is, $D = \{L_n, L_{n-1}, \ldots, L_1, R_1, R_2, \ldots, R_n\}$. For every t ($t = 1, 2, 3, \ldots$), let l_t be the number of the tth selected node by algorithm S_g, and X_t its state (e.g.,

$l_t = 17$ and $X_t = R_5$ means that the tth selected node is node number 17 whose front packet is destined to the node that is at distance 5 to its right (node 22)). Observe that l_t and X_t are random variables. Let T_t be the number of nodes between the tth and the $(t+1)$st selections. Hence, $T_t = l_{t+1} - l_t$. In the context of Markov processes, T_t is referred to as the tth transition time. Since the state of the nodes are mutually independent, it follows from Remark 3.1 that the process $\{(X_t, T_t), t \geq 1\}$ is a semi-Markov process.

To evaluate the node utilization of S_g, we have to find the transition probabilities from every state x to every state y, $P(x, y)$, and the expected transition time from every state x, $\bar{T}_x$. By definition,

$$\text{(3.1)} \qquad \begin{aligned} P(x, y) &= \Pr(X_{t+1} = y \mid X_t = x), \qquad x, y \in D, \\ \bar{T}_x &= E(l_{t+1} - l_t \mid X_t = x). \end{aligned}$$

Since $(l_{t+1} - l_t)$ assumes only values from 1 to $2n+1$, we have

$$\text{(3.2)} \qquad P(x, y) = \sum_{j=1}^{2n+1} \Pr(X_{t+1} = y, l_{t+1} - l_t = j \mid X_t = x).$$

Furthermore,

$$\text{(3.3)} \qquad \bar{T}_x = \sum_{y \in D} \sum_{j=1}^{2n+1} j \Pr(X_{t+1} = y, l_{t+1} - l_t = j \mid X_t = x).$$

Thus, in order to compute $P(x, y)$ and $\bar{T}_x$ we need the following probabilities:

$$\text{(3.4)} \qquad r(x, y, j) := \Pr(X_{t+1} = y, l_{t+1} - l_t = j \mid X_t = x).$$

$r(x, y, j)$ is the probability that the $(t+1)$st selected node has state y and is at distance j from the previous selected node l_t, given that l_t has state x. In the following technical lemmas, whose proofs are given in the Appendix, we derive these probabilities. For simplicity assume that $p = \frac{1}{2}$. For a general p, the evaluation technique is similar. Let,

$$\text{(3.5)} \qquad r(j) = r(L_n, R_n, j).$$

First, we explicitly derive $r(j)$ and then we express $r(x, y, j)$ in term of $r(j)$.

Lemma 3.1.

$$r(i) = \begin{cases} \dfrac{(2n-1)!}{(2n-i)!}\left(\dfrac{1}{2n}\right)^{i} & \text{if } 1 \leq i \leq n+1, \\[2ex] \dfrac{(2n-1)!}{(2n+1-i)!}\,\dfrac{1}{2}\left(\dfrac{1}{2n}\right)^{i-1} & \text{if } n+2 \leq i \leq 2n+1. \end{cases}$$

The proof is given in the Appendix.

In the next lemma we express $r(x, y, j)$ as a function of $r(j)$.

LEMMA 3.2. *For every* i, m, $1 \le i, m \le n$,

$$r(L_m, R_i, j) = \begin{cases} r(j)/2n \cdot r(n+1-m) & \text{if } n+1-\min\{m, i\} \le j \le 2n+1, \\ 0 & \text{if } j < n+1-\min\{m, i\}, \end{cases}$$

$$r(L_m, L_i, j) = \begin{cases} r(j)/2n \cdot r(n+1-m) & \text{if } n+1+i \le j \le 2n+1, \\ 0 & \text{if } j < n+1+i, \end{cases}$$

$$r(R_m, L_i, j) = \begin{cases} r(j)/2n \cdot r(n+1+m) & \text{if } n+1+\max\{m, i\} \le j \le 2n+1, \\ 0 & \text{if } j < n+1+\max\{m, i\}, \end{cases}$$

$$r(R_m, R_i, j) = \begin{cases} r(j)/2n \cdot r(n+1+m) & \text{if } n+1+m \le j \le 2n+1, \\ 0 & \text{if } j < n+1+m. \end{cases}$$

The proof is given in the Appendix.

The transition probabilities $P(x, y)$ are a direct consequence of (3.2), (3.4), and Lemma 3.2.

LEMMA 3.3. *For every* i, j, $1 \le i, j \le n$,

$$P(L_i, R_j) = \frac{1}{2n \cdot r(n+1-i)} \sum_{k=n+1-\min\{i,j\}}^{2n+1} r(k),$$

$$P(L_i, L_j) = \frac{1}{2n \cdot r(n+1-i)} \sum_{k=j+1+n}^{2n+1} r(k),$$

$$P(R_i, L_j) = \frac{1}{2n \cdot r(n+1+i)} \sum_{k=n+1+\max\{i,j\}}^{2n+1} r(k),$$

$$P(R_i, R_j) = \frac{1}{2n \cdot r(n+1+i)} \sum_{k=n+1+i}^{2n+1} r(k).$$

From Lemmas 3.3 and 3.1 we can compute $P(x, y)$. To compute $\bar{T}_x$ we need the following definitions. Let $q(k) = k \cdot r(k)$ and $Q(x, y)$ be defined as $P(x, y)$ but the $r(k)$'s are replaced with the $q(k)$'s. From the definition of $r(k)$ and (3.1) we obtain the following lemma.

LEMMA 3.4. *For every* $x \in D$, $\bar{T}_x = \sum_{y \in D} Q(x, y)$.

Now we are ready to evaluate $U_{S_g}(n, \frac{1}{2})$. Let x_0 be a given state in D, and $N^n_{y,x_0}(\tau^n_{y,x_0})$ be the number of nodes (number of selected nodes) elapsed in the network from the first selected node, until (and not including) the first time S_g selects a node whose state is x_0, given that the first selected node has a state y.

From the theory of Markov chains it is well known that

$$U_{S_g} = \frac{E(\tau^n_{x_0,x_0})}{E(N^n_{x_0,x_0})}. \tag{3.6}$$

Moreover, $\{\tau^n_{y,x_0}, N^n_{y,x_0} \mid y \in D\}$ are the solutions of

$$(I - \tilde{P})N^n_{x_0} = \bar{T}, \qquad (I - \tilde{P})\tau^n_{x_0} = 1, \tag{3.7}$$

where I is the $2n \times 2n$ identity matrix, $\tilde{P}$ is the matrix $P = [P(x, y)]$ in which the column corresponding to x_0 is replaced by a column of zeros, and $N^n_{x_0}$, $\tau^n_{x_0}$, $\bar{T}$, 1 are column vectors whose values are $(N^n_{y,x_0} \mid y \in D)$, $(\tau^n_{y,x_0} \mid y \in D)$, $(\bar{T}_y \mid y \in D)$, and $(1, 1, \ldots, 1)$, respectively.

By solving the equations in (3.7) we derive the *node utilization* of the greedy algorithm for various values of hearing ranges n. These results along with the upper bound from Theorem 2.1 and the *node utilization* of S^n are given in Table 3.1. Note that from [JaRo2], the optimal node utilization for $n = 1$ is 0.4043, so we can actually replace it with the bound of 0.5. Since S_g is better than S^n for small n's, we may be interested if S_g is also asymptotically optimal. In the next subsection we show that this is not the case.

3.2. The Utilization for Large n's. In the previous subsection we precisely analyzed the greedy algorithm. Unfortunately, this does not adequately describe the asymptotic behavior of the algorithm. In this subsection we show that the greedy algorithm is not asymptotically optimal by bounding its node utilization from above.

Table 3.1. Upper bounds and node utilizations.

Hearing range	Asymptotically optimal	Greedy algorithm	Upper bound
1	0.3333	0.3929	0.5000
2	0.2222	0.2503	0.3333
3	0.1667	0.1857	0.2500
4	0.1333	0.1481	0.2000
5	0.1111	0.1233	0.1667
6	0.0962	0.1057	0.1429
7	0.0847	0.0925	0.1250
8	0.0759	0.0823	0.1111
9	0.0690	0.0741	0.1000
10	0.0632	0.0674	0.0909
11	0.0583	0.0618	0.0833
12	0.0543	0.0571	0.0769
13	0.0507	0.0530	0.0714
14	0.0476	0.0495	0.0667
15	0.0449	0.0464	0.0625
16	0.0426	0.0437	0.0588
17	0.0404	0.0413	0.0556
18	0.0385	0.0391	0.0526
19	0.0367	0.0372	0.0500
20	0.0352	0.0354	0.0476
21	0.0337	0.0338	0.0455
22	0.0324	0.0324	0.0435
23	0.0312	0.0310	0.0417

To bound U_{S_g} from above, we consider the semi-Markov chain $\{(X_t, T_t), t \geq 1\}$ only at odd steps $t = 1, 3, 5, \ldots$. The new semi-Markov chain is defined by the two-step transition probability matrix P^2, where P is the one-step transition probability matrix. Let $T_x^2 = (l_{t+2} - l_t \mid d(l_t) = x)$ be the two-step transition distance given that the state of the tth selected node is x, and $\bar{T}_x^2 = E(l_{t+2} - l_t \mid d(l_t) = x)$ is its expectation. Also, let $(\mu_x \mid x \in D)$ be the stationary probability distribution of the states of $\{X_t \mid t \geq 1\}$.

Since $\{X_t \mid t \geq 1\}$ is irreducible and aperiodic, $(\mu_x \mid x \in D)$ is also the stationary probability distribution of the Markov chain governed by P^2. Therefore, we have

$$U_{S_g} = \frac{1}{\sum_{x \in D} \mu_x \bar{T}_x} = \frac{2}{\sum_{x \in D} \mu_x \bar{T}_x^2}. \tag{3.8}$$

The equalities in (3.8) are intuitively clear. Indeed, observe that $\sum_{x \in D} \mu_x \bar{T}_x$ is the long-run average number of nodes between two consecutive selections of S_g. Similarly, $\sum_{x \in D} \mu_x \bar{T}_x^2$ is the long-run average number of nodes between two consecutive odd selections. Formally, this result is a consequence of the mean ergodic theorem.

We bound U_{S_g} from above by bounding $\sum_{x \in D} \mu_x \bar{T}_x^2$ from below. The reason that we are bounding this expression and not $1/(\sum_{x \in D} \mu_x \bar{T}_x)$ is that it is impossible to find a bound to the latter that is lower than the one given in Theorem 2.1.

For simplicity of notation, we denote the states $L_n, L_{n-1}, \ldots, L_1, R_1, R_2, \ldots, R_n$ by $1, 2, \ldots, 2n$, respectively. The method that we use to derive our bound is to bound $\sum_{j=i}^{2n} \mu_j$ and $\bar{T}_k^2$ from below. From Lemma 3.3 it follows that, for every i_0 and $i > j$,

$$P(i_0, i) \geq P(i_0, j). \tag{3.9}$$

This is also intuitively clear, since if a given node is selected by S_g when it is at state j, it would have been selected if it were at state i.

From the definition of the stationary probabilities and (3.9), it follows that, for $i > j$,

$$\mu_i = \sum_{k=1}^{2n} \mu_k P(k, i) \geq \sum_{k=1}^{2n} \mu_k P(k, j) = \mu_j. \tag{3.10}$$

This implies that

$$\sum_{j=i}^{2n} \mu_j \geq \frac{2n - i + 1}{2n}, \qquad i = 1, 2, \ldots, 2n. \tag{3.11}$$

Otherwise, at least one μ_j with $j \geq i$ has $\mu_j < 1/2n$ and one μ_j with $j < i$ has $\mu_j > 1/2n$, which contradicts (3.10).

As to the lower bound on $\bar{T}_k^2$, clearly, $\bar{T}_k^2$ is at least $2n$ plus something that increases linearly with k. This follows from the fact that, for every given distance, the number of states that can be legally selected, decreases linearly with k. Also, since $p=\frac{1}{2}$, the slope is expected to be $\frac{1}{2}$. This result is formally proven in the next lemma.

LEMMA 3.5. *For every* $1 \le k \le 2n$, $\bar{T}_k^2 \ge 2n + k/2$.

The proof is given in the Appendix.

Inequality (3.11) implies the following lemma whose proof is also given in the Appendix.

LEMMA 3.6. $\sum_{k=1}^{2n} (k/2)\mu_k \ge 0.25 + 0.5n$.

The proof is given in the Appendix.

Now we have all the technical details to show that S_g is not asymptotically optimal.

THEOREM 3.1. $U_{S_g} \le 1/(1.25n + 0.125)$ (*i.e.*, S_g *is not asymptotically optimal*).

PROOF. From (3.8) and Lemmas 3.5 and 3.6,

$$U_{S_g} \le \frac{2}{\sum_{k=1}^{2n} (2n + k/2)\mu_k} \le \frac{2}{2n + 0.5n + 0.25} = \frac{1}{0.125 + 1.25n}. \qquad \square$$

4. An Improved Upper Bound on the Optimal Algorithm. From Table 3.1 we see that, for small n's ($n \le 21$), the *node utilization* under S_g is larger than that of S^n. For $n = 1$ the *node utilization* of S_g is 18% larger than that of S^n and only 3% smaller than that of the optimal algorithm from [JaRo2]. However, for $n > 1$ the *node utilization* of both S^n and S_g are far away from our naive upper bound of $1/(n+1)$. In this section we improve the upper bound for small n's. Note that from the asymptotic optimality of S^n, we cannot improve the upper bound very much for large n's.

Let n be a given hearing range. To derive an upper bound we relax the constraints on the transmission interference. Let C be the set of legal selections as defined in Section 2. From the definition, $S \in C$ if and only if, for every two nodes $i < j$ selected by S,

$$d(j) - i > n \quad \text{and} \quad j - d(i) > n. \tag{4.1}$$

For every $k \ge 1$ define the following sets of C_k*-legal selection* algorithms. Let $\mathbf{d}$ be a given system state and S a selection algorithm. Furthermore, let $l_0 < l_1 < l_2 < \cdots$ be the numbers of the nodes selected by S when the system is at state $\mathbf{d}$.

A C_k-legal selection is similar to a legal selection except for the following. Every kth selected node is permitted to interfere with the transmission of the previous selected node, but not vice versa. Formally, this notion is defined below.

DEFINITION 4.1. The selection S is C_k-legal if, for every $i \geq 1$,

$$d(l_{ik}) - l_{ik-1} > n,$$

and, for $j=(i-1)k, \ldots, ik-2$,

$$d(l_{j+1}) - l_j > n \quad \text{and} \quad l_{j+1} - d(l_j) > n.$$

For every k let

$$U^k(n, \tfrac{1}{2}) = \sup_{S \in C_k} U_S(n, \tfrac{1}{2}).$$

LEMMA 4.1. *For every* k, $U^*(n, \frac{1}{2}) \leq U^k(n, \frac{1}{2})$.

PROOF. From (4.1) and Definition 4.1, $C \subseteq C_k$, which provides the lemma. □

Exploring the sequence of the bounds $U^k(n, \frac{1}{2})$ is beyond the scope of this paper, but we conjecture that it is decreasing with k and approaches $U^*(n, \frac{1}{2})$ as $k \to \infty$. From Lemma 4.1, any $U^k(n, \frac{1}{2})$ that we can evaluate will serve as another upper bound to $U^*(n, \frac{1}{2})$. This is done below.

From the definition of a C_k-legal selection, we would expect that a selection that consecutively selects sets of k nodes for which the maximal distance between the two extreme nodes is minimal, is optimal in C_k if the selection is C_k-legal. This selection is formally defined below, and is shown to be optimal in C_k.

DEFINITION 4.2. For every system state $\mathbf{d}$, the selection algorithm S_{g_k} scans the tandem from left to right and selects the nodes as follows. The first k selected nodes, $l_0, l_1, \ldots, l_{k-1}$, are the first k legal selections with a minimal l_{k-1}. The rest of the selections are defined recursively. Suppose that S_{g_k} already selected i sets of nodes, each of which contains k selected nodes. The $(i+1)$st set of nodes selected by S_{g_k} are $l_{ik} < l_{ik+1} < \cdots < l_{(i+1)k-1}$ which minimizes $l_{(i+1)k-1} - l_{ik-1}$ and satisfies

$$d(l_{ik}) - l_{ik-1} > n,$$

$$d(l_{j+1}) - l_j > n, \qquad l_{j+1} - d(l_j) > n \qquad (j = ik, \ldots, (i+1)k-2).$$

Clearly, S_{g_k} is C_k-legal and makes the selection of every set of k nodes so as to minimize the number of nodes that are not permitted to transmit in the current set. Note that no consideration is given to the nodes to be selected later in the process. However, since we relax the constraint on every kth selection, it turns out, in the next lemma, that this selection is optimal in C_k.

LEMMA 4.2. *For every k, $U^k(n, \frac{1}{2}) = U_{S_{g_k}}(n, \frac{1}{2})$.*

PROOF. For every $S \in C_k$ and system state $\mathbf{d}$, let $l_{i\cdot k}(\mathbf{d}, S)$ be the number of nodes in the network, from the left edge node until and including the $(i\cdot k)$th selected node, given that the network is at state $\mathbf{d}$ and S is used. Since the interselection distance for any maximal S in C_k is bounded (by $4n+1$), we have

$$U_S(\mathbf{d}) = \limsup_{i\to\infty} \frac{i}{l_i(\mathbf{d}, S)}.$$

Moreover, since the interselection distance between the $i(k+1)$st and the ikth selected nodes is bounded too (by $(4n+1)k$), we also have

$$U_S(\mathbf{d}) = \limsup_{i\to\infty} \frac{i\cdot k-1}{l_{i\cdot k-1}(\mathbf{d}, S)}. \tag{4.2}$$

Therefore, from (2.2) it is sufficient to show that, for every i, $\mathbf{d}$, and $S \in C_k$,

$$l_{i\cdot k-1}(\mathbf{d}, S) \geq l_{i\cdot k-1}(\mathbf{d}, S_{g_k}). \tag{4.3}$$

We prove (4.3) by induction on i. Let $S \in C_k$ and $\mathbf{d}$ be a given system state. For $i=1$ we have $l_k(\mathbf{d}, S_{gk}) \leq l_k(\mathbf{d}, S)$ by Definition 4.2. Assume that (4.3) is satisfied for all j, $j\cdot k \leq i\cdot k$. The $(i+1)$st set of nodes selected by S, $l_{ik}(\mathbf{d}, S) < l_{ik+1}(\mathbf{d}, S) <$ $\cdots < l_{(i+1)k-1}(\mathbf{d}, S)$, satisfies the constraints of Definition 4.1. These selected nodes are also C_k-legal as the $(i+1)$st set of nodes, given that the nodes selected so far are those nodes selected by S_{gk}. Indeed, by the induction assumption

$$d(l_{ik}(\mathbf{d}, S)) - l_{ik-1}(\mathbf{d}, S_{gk}) \geq d(l_{ik}(\mathbf{d}, S)) - l_{ik-1}(\mathbf{d}, S) > n.$$

Hence, $l_{ik}(\mathbf{d}, S)$ is C_k-legal for S_{gk}. The rest of the nodes selected by S are clearly C_k-legal for S_{gk}, since the interference constraints for them are the same for both algorithms. Thus, from Definition 4.2

$$l_{(i+1)k-1}(\mathbf{d}, S_{g_k}) \leq l_{ik-1}(\mathbf{d}, S_{g_k}) + l_{(i+1)k-1}(\mathbf{d}, S) - l_{ik-1}(\mathbf{d}, S_{g_k}) = l_{(i+1)k-1}(\mathbf{d}, S). \qquad \square$$

Lemmas 4.1 and 4.2 provide the tools for obtaining a better upper bound for the maximal *node utilization*. The *node utilization* of S_{g_k} is evaluated in a similar way to that of S_g. It is easy to verify that under S_{g_k} every kth selected node is a renewal point in the selection process. Moreover, every selection of a set of k nodes is independent of the location and the state of the previous selected nodes (including the last selected node).

Let $W_k(\mathbf{d}) = l_{(i+1)k-1}(\mathbf{d}, S_{g_k}) - l_{ik-1}(\mathbf{d}, S_{gk})$ and $\bar{W}_k = E(W_k(\mathbf{d}))$, where the expectation is taken with respect to the distribution of the system state $\mathbf{d}$. Since there are k selected nodes between two consecutive renewal points we have from well-known results in renewal theory

$$U_{S_{g_k}} = \frac{k}{\bar{W}_k}.$$

To find $\bar{W}_k$ we have to compute the distribution of the minimal number of nodes elapsed in a selection process, until k nodes are selected without interfering with each other's transmission, given that the last selected node among the previous selection has state L_n. We do not have a fast algorithm to find this distribution, however, since the number of possible states is bounded, the computation is feasible for small k's and n's.

To demonstrate the improvement of the bound, relative to $1/(n+1)$, we compute it for $n=2$ and $k=2$. By simple case analysis we found that $\Pr(W_2=6)=0.4873$, $\Pr(W_2=7)=0.3307$, $\Pr(W_2=8)=0.1022$, $\Pr(W_2=9)=0.0712$, and $\Pr(W_2=10)=0.0086$. This yields an upper bound of $U_{S_{g_2}}=2/\bar{W}_2=0.2948$ compared with the naive bound of $\frac{1}{3}$. The bound can be improved by taking a larger k.

5. Discussion. In this section we briefly discuss a few aspects regarding our model and analysis.

5.1. Algorithm Implementations. This paper has concentrated on the calculation of the node utilization under the asymptotically optimal and the greedy algorithms. All the algorithms can easily be implemented in a central manner. However, to determine the transmitting nodes in a distributed manner is somewhat less practical. A node cannot decide whether to transmit until the protocol decision that is propagating from left to right reaches that node.

Nevertheless, we may approximate S^n and S_g by prepartitioning the nodes into disjoint intervals of size I and use the following version of our protocols. The selections within an interval are made independently of the selections at other intervals. Moreover, within each interval, all selections, except for the one furthest left, are determined as in the original protocol. In selecting the one furthest left, the revised protocol assumes that the selected node furthest right at the preceding interval is at state R_n. The new protocols are legal, and approximate the original ones if I is not too small. Now, in order to determine the nodes that transmit, the protocol can use reservation subslots and be executed independently within each interval. This procedure continues in phases, where each phase corresponds to a slot in our model. To achieve a fair version of S^n, we can alternate among the state sets $R^{j\cdot k}$, $L^{j\cdot k}$, on a phase basis.

5.2. A model with Memory. Our memoryless model is used only to approximate a realistic system, in which the system state at slot $t+1$, $\mathbf{d}_{t+1}$ depends on the system state at slot t, $\mathbf{d}_t$. It is of interest to see whether this dependency has a negative impact on the node utilization. For this purpose we carried out a simulation of a model with $n=1$ and where a node that does not transmit remains with its previous message. It turned out that the optimal selection algorithm (see [JaRo2]) in the simulation gave a node utilization of 0.4015, whereas in the memoryless model analysis it gave 0.4043. These results are reported in [JaRo1], and make us believe that a memoryless model could be a quite practical assumption.

Appendix

THE PROOF OF LEMMA 3.1. First we show that, for $p = \frac{1}{2}$ and every n,

(A.1a) $r(1) = \frac{1}{2}n$, $\quad r(j+1) = r(j)(1-(j/2n)) \quad$ for $1 \le j \le n$,

(A.1b) $r(n+2) = r(n+1)(1-\frac{1}{2})$, $\quad r(j+1) = r(j)(1-((j-1)/2n))$ for $n+2 \le j \le 2n$.

To compute $r(j)$, note that the conditional events $\{X_{t+1} = R_n,\ l_{t+1} - l_t = j \mid X_t = L_n\}$ depends on the states of nodes k, $d(k)$, $k > l_t$. More precisely, the $(t+1)$st selected node, l_{t+1}, could be any node $l_t + i$, $1 \le i \le 2n+1$, and the selection depends on the states $d(k)$, $l_t < k \le l_t + 2n + 1$. The probability that we are looking for, $r(j)$, is the probability that $l_{t+1} = l_t + j$ and $d(l_t + j) = R_n$, given that $X_t = d(l_t) = L_n$.

Note that

$$\{X_{t+1} = R_n,\ l_{t+1} - l_t = j \mid X_t = L_n\} = \{l_{t+1} = l_t + j,\ d(l_t + j) = R_n \mid X_t = L_n\}.$$

This event occurs if and only if node $l_t + j$ can be legally selected, $d(l_t + j) = R_n$, and a selection of any node between l_t and $l_t + j$ is not legal.

To find the probability of the event above, we need the following sets of states. For $1 \le k \le n$ let $\Phi^k = \{R_n, \ldots, R_{n-k+1}\}$; and for $n < k \le 2n$ let $\Phi^k = \{R_n, \ldots, R_1, L_1, \ldots, L_{k-n}\}$. Now, given that $X_t = L_n$, the next legal selection of S_g is $l_t + j$ (for a given j), if it is the first node among nodes $l_t + i$, $i \ge 1$, that satisfies

$$d(l_t + i) \in \Phi^j \qquad \text{if} \quad 1 \le j \le n,$$

or

$$d(l_t + i) \in \Phi^n \qquad \text{if} \quad j = n+1,$$

or

$$d(l_t + i) \in \Phi^{j-1} \qquad \text{if} \quad n+2 \le j \le 2n+1.$$

From (3.5) it follows that $r(1) = \Pr(d(l_{t+1}) = R_n) = 1/2n$. For $1 \le j \le n$,

(A.2)
$$\begin{aligned} r(j+1) &= \Pr(d(l_t + i) \in \Phi^i,\ 1 \le i \le j;\ d(l_t + j + 1) = R_n) \\ &= \Pr(d(l_t + i) \in \Phi^i,\ 1 \le i \le j-1;\ d(l_t + j + 1) = R_n) \cdot \Pr(d(l_t + j) \in \Phi^j) \\ &= \Pr(d(l_t + i) \in \Phi^i,\ 1 \le i \le j-1;\ d(l_t + j + 1) = R_n) \cdot (1-(j/2n)) \\ &= r(j)(1-(j/2n)). \end{aligned}$$

The last equality follows from the independence among the nodes' state and the uniform distribution of the node's state on $\{R_1, \ldots, R_n, L_1, \ldots, L_n\}$. For $n+2 \le j \le 2n$, the proof of (A.1) is similar.

Now the lemma immediately follows from the recursive relations in (A.1). □

THE PROOF OF LEMMA 3.2. We want to express $r(x, y, j)$ as a function of $r(j)$. Let $1 \le i \le n$. If $1 \le j < n+1-i$, then $r(L_n, R_i, j) = 0$. If $n+1-i \le j \le n$, then

(A.3)
$$r(L_n, R_i, j) = \Pr(d(l_t + k) \in \Phi^{n-k+1},\ 1 \le k \le j-1,\ d(l_t + j) = R_i) = r(j).$$

Furthermore, for every $m \le n-1$, if $j < n+1-\min\{i, m\}$, then $r(L_m, R_i, j) = 0$. If $n+1-\min\{i, m\} \le j \le n+1$, then

$$\text{(A.4)}\quad r(L_m, R_i, j) = \Pr(d(l_t+k) \in \Phi^{n-k+1}, n+1-m \le k \le j-1; d(l_t+j) = R_i)$$
$$= \frac{r(L_{m+1}, R_i, j)}{\Pr(d(l_t+n-m) \in \Phi^{m+1})} = \frac{r(L_n, R_i, j)}{\prod_{k=1}^{n-m} \Pr(d(l_t+k) \in \Phi^{n-k+1})}$$
$$= \frac{r(j)}{2n \cdot r(n-m+1)}.$$

The second equality is obtained by multiplying the numerator and the denominator by $\Pr(d(l_t+n-m) \in \Phi^{m+1})$ and using the independence among the states of the nodes. The third equality is obtained by recursively using the same argument. The last equality follows from (A.3) and the representation of $r(j)$ in (A.1).

A similar computation is made for the rest of the $r(x, y, j)$'s. □

THE PROOF OF LEMMA 3.5. First consider the case that the transmitter has state $k \in \{1, 2, \ldots, n\}$, i.e., a state of the form L_i, where $i = n-k+1$. The next transmitter will be $n-i+j$ nodes later for some $j = 1, 2, \ldots, n+i+1$. From Lemma 3.2 it follows that, for a given distance $n-i+j$:

(i) If $j \le i$, then the next transmitter will be at distance $n-i+j$ and have state $R_{i+1-j}, \ldots, R_n$ with equal probability (i.e., $r(L_i, y, n-i+j)$, $y \in \{R_{i+1-j}, \ldots, R_n\}$ are all equal).
(ii) If $j = i+1$, then the next transmitter will be at distance $n-i+j$ and have state $R_1, \ldots, R_n$ with equal probability.
(iii) If $j > i+1$, then the next transmitter will be at distance $n-i+j$ and have state $L_{j-(i+1)}, \ldots, L_1, R_1, \ldots, R_n$ with equal probability.

Given that case (i) holds, then T_k^2 is at least $(n-i+j)+(i+1-j)+(n+1), \ldots, (n-i+j)+(n)+(n+1)$ with equal probability. This yields an average of at least $2n+1+(n-i+j+1)/2 > 2n+k/2$.

Given that case (ii) holds, then T_k^2 is at least $(n-i+j)+(1)+(n+1), \ldots, (n-i+j)+(n)+(n+1)$ with equal probability. This yields an average of at least $2n+2+(n+1)/2 > 2n+k/2$.

Finally, given that case (iii) holds, then T_k^2 is at least $(n-i+j)-(j-(i+1))+(n+1), \ldots, (n-i+j)-(1)+(n+1)$ (for the L's) or $(n-i+j)+(1)+(n+1), \ldots, (n-i+j)+(n)+(n+1)$ (for the R's) with equal probability. This yields an average of at least

$$2n+j-i+1+\frac{n+j-i}{n+j-i-1}\,\frac{n-j+i+1}{2} > 2n+\frac{k}{2}.$$

Since under any conditional distance of the first transmitter, T_k^2 is at least $2n+k/2$ we have the lemma for $k = 1, \ldots, n$. Similarly, if $k \in \{n+1, \ldots, 2n\}$, i.e., a state of the form R_i with $i = k-n$, we again consider the next two transmitters.

The next transmitter will be at distance $n+i+j$, and then have state $L_{i+j-1}, \ldots, L_1, R_1, \ldots, R_n$ with equal probabilities. Similar to the above, T_k^2 is at least $(n+i+j)+(n+1)-(i+j-1), \ldots, (n+i+j)+(n+1)-1$ (for the L's) or $(n+i+j)+(n+1)+1, \ldots, (n+i+j)+(n+1)+n$ (for the R's) with equal probabilities. This yields an average of at least

$$2n+i+j+1+\frac{n+j+i}{n+j+i-1}\,\frac{n-i-j+1}{2}>\frac{5n+i+j+3}{2}>2n+\frac{k}{2}. \qquad \square$$

The Proof of Lemma 3.6. We show by induction that, for every i,

$$\sum_{k=i}^{2n}\frac{k}{2}\mu_k \geq \sum_{k=i}^{2n}\frac{k}{2}\frac{1}{2n}+\frac{i}{2}\left(\left(\sum_{k=i}^{2n}\mu_k\right)-\frac{2n-i+1}{2n}\right). \tag{A.5}$$

For $i=2n$ we have an equality in (A.5). Assume that (A.5) holds for $i=2n, \ldots, j+1$. For $i=j\geq 1$ we have, by the induction assumption and (3.11),

$$\begin{aligned}\sum_{k=j}^{2n}\frac{k}{2}\mu_k &= \frac{j}{2}\mu_j+\sum_{k=j+1}^{2n}\frac{k}{2}\mu_k \geq \frac{j}{2}\mu_j+\sum_{k=j+1}^{2n}\frac{k}{2}\frac{1}{2n}+\frac{j+1}{2}\left(\left(\sum_{k=j+1}^{2n}\mu_k\right)-\frac{2n-j}{2n}\right)\\ &\geq \frac{j}{2}\mu_j+\sum_{k=j+1}^{2n}\frac{k}{2}\frac{1}{2n}+\frac{j}{2}\left(\left(\sum_{k=j+1}^{2n}\mu_k\right)-\frac{2n-j}{2n}\right)\\ &= \sum_{k=j}^{2n}\frac{k}{2}\frac{1}{2n}+\frac{j}{2}\left(\left(\sum_{k=j}^{2n}\mu_k\right)-\frac{2n-j+1}{2n}\right),\end{aligned}$$

which provides (A.5). Now, the lemma follows from (A.5) by taking $i=1$. $\square$

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