where $\eta(u)$ is an arbitrary variation of $f(x)$ and λ_1, λ_2 are Lagrange multipliers. Thus, $f(x)$ must satisfy

$$2\int_0^\infty f(y+u)\,\ln\left[\int_0^\infty f(y+v)f(v)\,dv\right]dy + 2\int_0^u f(u-y)\,\ln\left[\int_0^\infty f(y+v)f(v)\,dv\right]dy = \lambda_1 u^n+\lambda_2-2; \qquad \text{for all } u \geqslant 0. \tag{25}$$

It follows that $f(x)$ is the solution of the nonlinear integral equation (25); the Lagrange multipliers λ_1, λ_2 are determined from the constraints $\int_0^\infty f(x)\,dx = 1$ and (23) once the general solution of (25) is found. This, however, appears to be very complex to obtain analytically.

We now show that under a first-order moment constraint [$n = 1$ in (23)], the probability density $f(x)$ satisfying (25) *cannot* be exponential, in contrast to the result in Subsections A and B. Indeed, suppose $f(x) = \beta^{-1}e^{-x/\beta}$; then we can evaluate the sum of the two integrals on the left side of (25), and it can be seen that it is not a first-order polynomial in u. Specifically, the left side of (25) is equal to

$$2(1+\ln(1/2\beta)) - \frac{2u}{\beta} - 4e^{-u/\beta}; \qquad u \geqslant 0. \tag{26}$$

It would be of interest to solve (25) analytically for the timeout density $f(x)$ at least for the important case $n = 1$.

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Deterministic Routing to Buffered Channels

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Abstract—**Consider *n* exponential transmission channels which transmit information with different rates. Every channel has a buffer which is capable of storing an unlimited number of messages. A new message first arrives at the controller, which immediately routes it to one of the channels according to an infinite deterministic routing sequence. A cost per unit of staying time is charged in each of the channels (channel dependent cost), and the long-run average staying cost is taken as the cost criterion.**

For every *n* and a Poisson arrival process, a lower bound to the cost is found and a new routing policy, the *golden ratio policy*, is presented and its cost is evaluated. It is shown that for a variety of system parameters, the golden ratio routing policy has a cost close to the lower bound.

I. INTRODUCTION

Message routing in a communication network was extensively studied by several authors, e.g., [1]-[5], [13], [14].

In most studies, static routing policies were analyzed. By static policy one means a policy which routes an outgoing message from a node with fixed probability, independent of the past history of the entire network. Various algorithms have been proposed to find the optimal routing probabilities which minimize the long-run average delay per message [1], [4], [5], [14]. However, it is often the case that nonstatic routing policies achieve a smaller average delay. For an example of two parallel channels see [2]. For more general networks it appears very difficult to find the optimal policy, although it is clear that in most cases it is not a static one (see [13], [15]).

Dynamic routing was explored in [10], in which a conceptual form of an algorithm is given when the input to each node is constant in time.

In this paper we study a single node with several heterogeneous outgoing links under a subset of the distributed dynamic routing policies and a general cost criterion. The subset includes the optimal policy in the case where there are two channels. We use results from [6] to bound the optimal cost and matrix-geometric solutions [11] to evaluate the cost of a new routing policy.

Consider n parallel transmission channels which transmit information with different rates. That is, the transmission time of a message through a channel depends on the channel. Every channel has a buffer which is capable of storing an unlimited number of messages and can transmit at most one message at any moment of time. A message is transmitted without interruption. A new message first arrives at the controller, which immediately routes it to one of the channels. In a physical environment the controller is an intelligent multiplexer or a given transmission node in a network, and the channels are parallel computer components or a subset of outgoing links, respectively.

We assume that the controller routes the messages instantly and the messages are enqueued at the buffers of the channels. We further assume that there is no travel time between the controller and the channels.

The messages arrive at the controller as a Poisson process with rate λ. A message which is routed to a channel joins the end of the queue, or starts transmission immediately, if the buffer is empty.

The transmission time of a message through channel i is exponentially distributed with rate μ_i. All transmission times are assumed to be independent. Different transmission rates in a network spring from two reasons: 1) different capacities of the outgoing links, and 2) different remaining capacities due to other traffic in the network, when the original capacities are equal.

We also assume that the well-known necessary condition for ergodicity holds, that is,

$$\lambda < \sum_{i=1}^{n} \mu_i.$$

For practical reasons we are interested in routing policies

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which can be distributively implemented (i.e., the controller does not observe the contents of the buffers). Therefore, the well-known results on the optimality of the shortest queue rule are not of interest.

A general set of distributed routing policies is the following set, called *deterministic routing policies.* Suppose that a deterministic n-valued sequence $r = (r_1, r_2, \cdots)$, $1 \le r_k \le n$, is given and the messages are numbered according to their order of arrival. For every k, message k is routed to channel r_k. The sequence r will be referred to as the routing sequence.

To reflect the differences in the utility among the channels, we charge every message at channel i, a cost $C^{(i)}$ per unit of staying time. Let $V_k^{(i)}(r)$ be the total expected cost at channel i, until the kth arrival *at channel i,* using the routing sequence r. Define the total long-run average cost of using the routing sequence r

$$\bar{V}(r) = \liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{n} V_k^{(i)}(r). \tag{1.1}$$

Also let $\bar{V} = \inf_r \bar{V}(r)$. A routing sequence r^* is optimal if $\bar{V}(r^*) = \bar{V}$.

For $n = 2$, the structure of the optimal routing sequence was found in [7]. Independently, it was shown in an earlier version of this paper that the policy in [7] is ϵ-optimal for every ϵ.

In a previous study, [13], we considered a simpler model for the routing problem. In that model the channels are assumed to be slotted and messages are "packetized" and routed to one of the channels. The channels have no buffers and the total throughput is taken as the cost criterion. The *golden ratio routing* policy was proposed there, and it was shown that its throughput is within at least 98.4 percent of the upper bound. In this study we show that in a more realistic model, its average delay is also close to the lower bound. The model in this paper is far more complicated than the previous one and we are required to use a new technique—matrix geometric solutions [11]—to evaluate the proposed policy.

In Section II we give a lower bound to $\bar{V}$. In Section III we define and analyze the golden ratio routing policy and present its performance in various systems.

II. A LOWER BOUND

For a given routing sequence r let $N_j^{(i)}(r)$ be the number of messages in the buffer of channel i (including the one which is being transmitted) that are found by the jth arrival at that channel. Also, let

$$\bar{N}^{(i)}(r) = \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} E(N_j^{(i)}(r)).$$

Note that under stationary conditions, $\bar{N}^{(i)}(r)$ is the expected queue length found by an arrival at channel i. Hence, the expected staying time of a message in queue i is $\mu_i^{-1}(\bar{N}^{(i)}(r) + 1)$, and from Little's theorem the expected queue length at channel i is $\lambda p^{(i)} \mu_i^{-1}(\bar{N}^{(i)}(r) + 1)$. Here, $p^{(i)}$ is the proportion of messages that are routed to channel i under sequence r. It is easy to verify that under stationary conditions the limit in (1.1) exists and

$$\bar{V}(r) = \sum_{i=1}^{n} C^{(i)} \lambda p^{(i)} \mu_i^{-1} (\bar{N}^{(i)}(r) + 1).$$

Since we are mainly interested in the average delay we consider here the case where $C^{(i)} = 1/\lambda p^{(i)}$, which yields

$$\bar{V}(r) = \sum_{i=1}^{n} \mu_i^{-1} \bar{N}^{(i)}(r) + \sum_{i=1}^{n} \mu_i^{-1}.$$

Since the second summand is independent of the routing policy, we may exclude it from the cost criterion.

Let $0 \le p^{(i)} < 1$, and let $A_k^{(i)}(r)$ be the number of messages among the first k arrivals that are routed to channel i under r. It was shown by Hajek [6, sect. 5] that if

$$\liminf_{k\to\infty} \frac{A_k^{(i)}(r)}{k} \ge p^{(i)}$$

and

$$p^{(i)} < \min \{1, \mu_i/\lambda\}$$

then

$$\bar{N}^{(i)}(r) \ge \frac{x^{(i)}}{1 - x^{(i)}}$$

where $x^{(i)}$ is the unique solution of

$$\left[\frac{1}{1 + (1 - x^{(i)}) \mu_i/\lambda} \right]^{1/p^{(i)}} = x^{(i)}.$$

Moreover, if $p^{(i)} \ge \mu_i/\lambda$ then $\bar{N}^{(i)}(r) = \infty$.

It is well known (see, e.g., [9, p. 251]) that in a $GI/M/1$ system, where the arrival process is composed of every $1/p^{(i)}$ arrival (provided it is an integer) from a Poisson process, $x^{(i)}/(1 - x^{(i)})$ is the stationary expected number of messages in buffer i found by an arrival.

Since every message must be routed to one of the channels and μ_i is positive, a lower bound to $\bar{V}$ is obtained by solving the following extremum problem:

$$\min \sum_{i=1}^{n} \frac{\mu_i^{-1} x^{(i)}}{1 - x^{(i)}} \tag{2.1}$$

such that

$$0 \le x^{(i)} \le \min(1, 1/\alpha_i), \quad 0 \le g_i(x^{(i)}) < \alpha_i, \qquad \sum_{i=1}^{n} g_i(x^{(i)}) = 1 \tag{2.2}$$

where $\alpha_i = \mu_i/\lambda$ and $g_i(x) = -\ln(1 + (1 - x)\alpha_i)/\ln(x)$.

Since $g_i(x)$ increases in the interval $(0, \min\{1, 1/\alpha_i\})$, the set of feasible solutions is convex. Furthermore, the cost function is an increasing convex function. Thus, we can use the Langrangian multipliers technique to solve the $x^{(i)}$'s by equating to zero the first derivatives of the Lagrangian function. The uniqueness of the solution is guaranteed by the convex properties above. The solution is not expressed in a close form; however, by using Newton's algorithm one can efficiently solve the equations.

Let $x^{(i)*}$, $i = 1, \cdots, n$, be the optimal $x^{(i)}$'s and $p^{(i)*} = g_i(x^{(i)*})$ be the "desirable routing proportions."

If there is a routing policy which routes every $1/p^{(i)*}$ message to channel i, for every i, then the lower bound will be obtained. Unfortunately, an equally distance routing policy for all channels is almost never feasible (see [8]). An exception is the case where all the μ_i's are equal. In this case $p^{(i)*} = 1/n$ and the lower bound is obtained by the round-robin policy. In the next section we shall present a routing policy which approximates the optimal solution of (2.1) and (2.2) by routing almost every $1/p^{(i)*}$ message to channel i, for every i.

III. THE GOLDEN RATIO ROUTING

Let $p^{(i)}$, $i = 1, \cdots, n$, be any desirable proportions (e.g., the $p^{(i)*}$ from Section II). We define a routing policy which attempts to distribute the routings to each channel, in almost equal distance, and is based on the properties of the multiplicative hashing function with golden ratio multiplicand, $\varphi^{-1} = (\sqrt{5}-1)/2 \approx 0.6180339887$.

Let N and $N^{(i)}$; $i = 1, 2, \cdots, n$ be integers such that

$$\lfloor p^{(i)}N \rfloor \le N^{(i)} \le \lceil p^{(i)}N \rceil \quad \text{and} \quad \sum_{i=1}^{n} N^{(i)} = N$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer smaller (greater) than or equal to x.

Thus,

$$\lim_{N\to\infty} \frac{N^{(i)}}{N} = p^{(i)}.$$

Let frac $(y) = y - \lfloor y \rfloor$, $a_j =$ frac $(j\varphi^{-1})$, and $A_N = \{a_j | j = 0, \cdots, N-1\}$.

Consider all the arrivals in batches of N consecutive messages each. The tth smallest point of A_N is identified with the tth message of each batch.

Definition 3.1: The golden ratio policy, $\pi_{\mathrm{GR}(N)}$, is the policy which assigns to channel i the messages corresponding to the points

$$\left\{ a_j \,\middle|\, \sum_{m=1}^{i-1} N^{(m)} \le j < \sum_{m-1}^{i} N^{(m)} \right\}.$$

It will be convenient to identify the points 0 and 1, and thus the points a_j are distributed over a circle C.

For example, let $p^{(1)} \approx 1/2$, $p^{(2)} \approx 3/8$, $p^{(3)} \approx 1/8$, and $\Sigma p^{(i)} = 1$. Taking $N = 8$, $N^{(1)} = 4$, $N^{(2)} = 3$, and $N^{(3)} = 1$, $\pi_{\mathrm{GR}(8)}$ routes to channel 1 the messages corresponding to 0, φ^{-1}, frac $(2\varphi^{-1})$, and frac $(3\varphi^{-1})$; to channel 2 the messages corresponding to frac $(4\varphi^{-1})$, frac $(5\varphi^{-1})$, and frac $(6\varphi^{-1})$; and to channel 3 the messages corresponding to frac $(7\varphi^{-1})$. Thus, the golden ratio policy keeps routing every eight consecutive messages to the channels in the following cyclic order: "1, 2, 1, 3, 2, 1, 2, 1."

When all the $p^{(i)}$'s are equal, the extremum problem implies that $N = n$ and $N^{(i)} = 1$, in which case $\pi_{\mathrm{GR}(n)}$ is the round-robin policy.

To evaluate $\bar{V}(r)$ for the golden ratio sequence r we need $\bar{N}^{(i)}(r)$. By the method of phases (see [11, ch. 1]) the process $(N_k^{(i)}(r) | k \ge 1)$ can be embedded into a countable state time homogeneous Markov-chain and $\bar{N}^{(i)}(r)$ can be expressed as follows.

Let N, $N^{(i)}$, $i = 1, \cdots, n$, be given and let $d_j^{(i)}$, $j = 0, \cdots, N^{(i)} - 1$, be the number of arrivals between two successive routings to channel i in a given batch (including the message which is routed next to the channel). Clearly, the $d_j^{(i)}$'s are uniquely determined by $\pi_{\mathrm{GR}(N)}$ and they are independent of the particular batch (i.e., a periodic routing with period N).

The states of channel i between successive routings to this channel will be called phases and be denoted by j, $j = 0, \cdots, N^{(i)} - 1$.

Let $a_{j,[j+1]}^{(i)}(k)$ be the probability that k messages are transmitted by channel i during phase j (between the jth and the $[j+1]$th routings to this channel), given that the channel is busy during the entire phase. ($[j+1] = (j+1) \bmod N^{(i)}$.) Since the time required to observe k arrivals from a Poisson process has an Erlang distribution

$$a_{j,[j+1]}^{(i)}(k) = \int_0^\infty \frac{(\mu_i x)^k}{k!} e^{-\mu_i x} \cdot \frac{\lambda^{d_j^{(i)}} x^{d_j^{(i)}-1}}{(d_j^{(i)}-1)!} e^{-\lambda x}\, dx$$

$$= \binom{k+d_j^{(i)}-1}{k} \left(\frac{\mu_i}{\lambda}\right)^k \left(\frac{\lambda}{\lambda+\mu_i}\right)^{k+d_j^{(i)}}.$$

Also let $b^{(i)}{}_{j,[j+1]}(k) = 1 - \Sigma_{l=0}^{k} a^{(i)}{}_{j,[j+1]}(l)$.

$$A_i(k) = \begin{bmatrix} 0 & a_{1,2}^{(i)}(k) & 0 & \cdots & 0 \\ 0 & 0 & a_{2,3}^{(i)}(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & a_{N^{(i)}-1,0}^{(i)}(k) \\ a_{0,1}^{(i)}(k) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B_i(k) = \begin{bmatrix} 0 & b_{1,2}^{(i)}(k) & 0 & \cdots & 0 \\ 0 & 0 & b_{2,3}^{(i)}(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdots & b_{N^{(i)}-1,0}^{(i)}(k) \\ b_{0,1}^{(i)}(k) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Let R_i be the unique matrix which solves

$$R_i = \sum_{k=0}^{\infty} R_i^k A_i(k)$$

and let $B(R_i) = \Sigma_{k=0}^{\infty} R_i^k B_i(k)$. (See [11, ch. 1] for uniqueness.) From [11],

$$\bar{N}^{(i)}(r) = \xi_i R_i (I - R_i)^{-2} \mathbf{1},$$

where $\mathbf{1}$ is the column vector of ones, and ξ_i is the positive left invariant eigenvector of $B_i(R)$ [i.e., $\xi_i = \xi_i B_i(R)$] normalized by $\xi_i(I - R_i)^{-1}\mathbf{1} = 1$.

The matrices R_i can be efficiently computed by successive substitutions. For further computational procedures see [11, sect. 1.9].

To evaluate the cost function of the golden ratio routing policy, $\bar{V}(\mathrm{GR})$, we define the measure $f = (\bar{V}(\mathrm{GR}) - \mathrm{LB})/\mathrm{LB}$, where LB is the lower bound found in (2.1).

The values of f under various system parameters had been computed and the results are reported in details in the previous version of this note, [12]. Here, we summarize the main results.

First, note that without loss of generality we may assume that $\lambda = 1$, since the $\bar{N}^{(i)}(r)$'s are invariant under the transformation into a system with an arrival rate of 1 and transmission rates of μ_i/λ. Furthermore, from a large number of examples that we considered, it was found that for a given number of channels n, f is mainly affected by $\mu^{(1)} = 1/n\Sigma\mu_i := 1/n\rho$ and $\mu^{(2)} = 1/n\Sigma\mu_i^2$. Here, ρ is the system utilization which is determined by $\mu^{(1)}$, and $\mu^{(2)}$ measures the variance of the transmission rate.

For $n = 5$, f varies from 0.045 to 0.125 at $\rho = 0.35$; from 0.039 to 0.099 at $\rho = 0.50$; and from 0.037 to 0.066 at $\rho = 0.75$.

For $n = 15$, f varies from 0.017 to 0.207 at $\rho = 0.35$; from 0.014 to 0.116 at $\rho = 0.50$; and from 0.030 to 0.081 at $\rho = 0.75$.

For $n = 30$, f varies from 0.018 to 0.247 at $\rho = 0.35$; from 0.014 to 0.145 at $\rho = 0.50$; and from 0.042 to 0.066 at $\rho = 0.75$.

Recall that LB is almost never obtained by any routing sequence. As a general conclusion from the analysis of many cases, one may say that for $\rho \ge 0.5$ (in which case good routing policies are most important); the relative difference between the golden ratio routing policy and the lower bound mainly varies between 4 and 10 percent. Moreover, it decreases as ρ increases.

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Theory of an Imperfect PCM Encoder–Decoder Pair Using Half-Channel Measures

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Abstract—**A theory is presented for the quantization noise power and gain of a PCM channel in terms of static and dynamic errors in encoder decision levels and decoder output levels. Decision profiles are used in a statistical model for encoder imperfections, while conditional output distributions describe those of the decoder. The model describes the combined behavior of any encoder and decoder that have been characterized separately in terms of the CCITT-defined half-channel tests. A linear regression model is used and the noise powers and offsets are shown to obey addition laws, while the gains obey a multiplicative rule.**

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INTRODUCTION

Before the advent of PCM-based switching, it was adequate to treat a point-to-point PCM channel as a unit [1] when assessing gain and signal-to-noise ratio (SNR) performance. Current test recommendations [2] call for separate measurements of encoder and decoder performance in the form of half-channel gain and SNR values to ensure that any encoder and decoder switched together will perform satisfactorily. A theory is developed here for the gain and noise power produced by an arbitrary encoder–decoder pair having static and dynamic errors. The object is to relate the resulting gain and noise powers to the half-channel values measured according to the CCITT definition [2]. Send and receive filters are not considered.

Terminology

An encoder is termed *ideal* if the decision levels V_I, $I = 1, \cdots, N + 1$, comply exactly with the prescribed encoding law, while a *real encoder* is subject to static and dynamic decision level errors. An *ideal decoder* produces pulse amplitudes Y_I at its output according to

$$Y_I = 0.5(V_I + V_{I+1}) \qquad I = 1, \cdots, N. \tag{1}$$

A *real decoder* produces output amplitudes which differ from Y_I by static and dynamic errors. The theory is developed by considering the gain and noise power of four test configurations.

a) The full channel test (FCT) contains a *real encoder* and a *real decoder.*

b) The encoder-side-half-channel test (ESHCT) consists of a *real encoder-under-test* and an *ideal decoder.*

c) The decoder-side half-channel test (DSHCT) consists of an *ideal encoder* and a *real decoder-under-test.*

d) The reference channel (RC) contains only *ideal* units.

MODELS FOR ENCODER AND DECODER

Encoder Model

The model uses decision profiles (DP) equivalent to classification probabilities employed in [3]. The decision profile, $D_I(V)$, for the Ith decision level is defined as the probability that a sample at amplitude V will be classified as being above level V_I. Fig. 1 shows examples of decision profiles. The shapes of the decision profiles are related to the cumulative distributions of the equivalent amplitude errors in the sample-and-hold and the quantizer. As the time-varying component is traceable in part to the rate of change of the input signal, e.g., jitter and aperture effects [4], the decision profiles can be signal dependent but have definite forms for stationary signals. The virtual decision levels at the top and bottom of the range, i.e., those not implemented in practical converters, are handled by putting $D_{N+1}(V) = 0$ and $D_1(V) = 1$.

The quantizing window $W_I(V)$ for the Ith interval is derived from the decision profile as

$$W_I(V) = D_I(V)[1 - D_{I+1}(V)] \qquad I = 1, \cdots, N. \tag{2}$$

This is the probability that an input signal sample at amplitude V will be classified into interval I. For an ideal quantizer, the decision profile for level V_I is a unit step and the quantizing window is unity within the interval and zero elsewhere. The probability that a sample of an input signal with PDF $p_v(V)$ will be classified into interval I is

$$p_I^* = \int_{-\infty}^{\infty} W_I(V) p_v(V)\, dV. \tag{3}$$

It can be verified that the sum of the p_I^* is unity.