

Resequencing Delay and Buffer Occupancy Under the Selective-Repeat ARQ

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Abstract—Consider a communication network that regulates retransmissions of erroneous packets by a selective-repeat (SR) automatic repeat request (ARQ) protocol. Packets are assigned consecutive integers, and the transmitter continuously transmits them in order until a negative acknowledgment or a time-out is observed. The receiver, upon receipt of a packet, checks for errors and returns positive/negative acknowledgment (ACK/NACK) accordingly. Only packets for which either NACK or time-out have been observed are retransmitted. Under SR ARQ, the receiver accepts packets that are out of order and must store them temporarily if it has to deliver them in sequence. The resequencing buffer requirements and the resulting packet delay constitute major factors in overall system considerations. We derive the distributions of the buffer occupancy and the resequencing delay at the receiver under a heavy traffic situation. This enables the network designer to determine how much buffer capacity at the receiver will guarantee certain specified performance measures.

I. INTRODUCTION

THE PROCEDURES whereby computer communications networks preserve the integrity of data sent from a transmitter to a receiver over a noisy path are known as automatic repeat request (ARQ) protocols. These provide for the data to be sent by packets, each of which is encoded for error detection by the receiver. The packets that arrive at the transmitter are assigned consecutive numbers that identify them uniquely (referred to as identifiers). First transmissions of packets are done in increasing order of identifiers.

Based on the error detection results, positive/negative acknowledgments (ACK/NACK) are sent by the receiver over a feedback channel, arriving at the transmitter after a round-trip delay. The acknowledgments bear the identifiers of the packets they acknowledge. If no feedback message is received within a predetermined interval, the transmitter interprets this as a NACK and retransmits the packet. This event is referred to as a *time-out*.

The retransmission of erroneous packets depends on the particular ARQ protocol used. There are three basic ARQ schemes: stop-and-wait, go-back- N , and selective-repeat (SR) [2], [9], [12], [5]. In stop-and-wait, the transmitter

remains idle after a packet's transmission until ACK, NACK, or a time-out is observed. In the first case, it transmits a new packet whereas in the last two cases it retransmits the preceding packet. Under the go-back- N protocol, $1 \leq N \leq \infty$, during the round-trip delay, the transmitter may send up to $N-1$ other packets. When a NACK is received, the transmitter stops sending new packets, backs up to the negatively acknowledged packet and retransmits it and all subsequent $N-1$ packets. Upon detecting an error in a packet, the receiver stops accepting new packets until the erroneous one is correctly received. Note that go-back-1 is the stop-and-wait ARQ. We focus on the selective-repeat protocol, under which the transmitter continuously sends new packets and the receiver accepts every packet that arrives error-free. Upon receipt of a NACK, or in case of a time-out event, only the corresponding packet is retransmitted.

To maintain integrity, it is a common requirement in computer networks that the receiver send out/release packets (to the user, to the next node or to the upper layer of the network architecture) in their original order. Under the go-back- N protocol, packets that arrive out of order are ignored by the receiver which does not have to allocate any buffers to them. Under the selective-repeat ARQ, however, those packets must be stored in the receiver's buffers until they can be sent out in the original order. The buffer needed for this purpose is referred to as a resequencing buffer, and the time that packets spend there as resequencing delay.

There are several important performance measures associated with ARQ protocols: the throughput, the packet delay, buffer occupancy at the transmitter, and buffer occupancy at the receiver. In [4], the performance under go-back-1 was analyzed, and the buffer occupancy distribution obtained when the buffer capacity at the transmitter is unlimited. For a finite buffer capacity, the overflow probability was derived, from which optimal time-out values were obtained. In [7], [15], the distribution of the packet delay and the buffer occupancy at the receiver, under go-back- N and SR, were derived. A modified version of the go-back- N , the stutter go-back- N , was suggested in [14], for links with high error rate or long propagation delay. It has been shown that for low traffic, the average buffer occupancy under the modified version approaches the corresponding value of an "idealized" ARQ protocol. The throughput under several variants of the

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selective-repeat ARQ was studied in [8], and [16]. Other related studies are [11], [6], [13], [1].

In [3], the throughput under the selective-repeat ARQ protocol was shown to outperform that of the go-back- N over a wide range of error rates. As can be seen from the analysis in [3], performance under selective-repeat is strongly dependent on the buffer capacity at the receiver. If that capacity does not meet the demand arising from the packet arrivals and the error rate, performance deteriorates drastically. The results in [3] indicate that the SR ARQ is promising; however, one has to be very careful when designing the size of the resequencing buffer.

Since SR is the only ARQ protocol that requires the receiver to buffer packets, to evaluate its performance one has to consider the overall delay of a packet. This consists of two parts:

- 1) queueing delay at the transmitter (i.e., the time between the packet's arrival and its successful transmission), and
- 2) resequencing delay at the receiver.

The goal of this paper is to evaluate the resequencing-buffer requirement and the resequencing delay under SR ARQ. (The buffer requirement and the queueing delay at the transmitter were studied before in [1], [7], [15].) For this purpose, we assume a heavy traffic situation, in which packets arrive at the transmitter from an infinite source. For any arrival process, this model places upper bounds on the resequencing-buffer requirements and resequencing delay.

Observe that the heavy traffic model is mainly important at the network design stage, when the statistics of the arrival process are not yet available. Thus the model enables the designer to specify the resequencing-buffer capacity and predict the worst-case delay without being restricted to performance degradation as indicated in [3].

In Section II we formulate the model, and in Section III we derive the probability generating function of the buffer occupancy, its first two moments, and the resequencing delay distribution of an arbitrary packet. In Section IV we obtain a tight approximation for the buffer occupancy distribution function; in Section V we present some numerical examples.

II. THE MODEL FORMULATION

Consider a pair of nodes, a transmitter and a receiver, which communicate data packets through a noisy channel, and ACK's/NACK's over a noiseless feedback channel. Packets that arrive at the transmitter are assigned consecutive integers that serve as their identifiers. We assume that the transmitter and the receiver have unlimited buffer capacity and that they regulate the retransmission of erroneous packets according to the following selective-repeat ARQ protocol.

The transmitter continuously transmits new packets in increasing numerical order as long as ACK's are received for the transmitted packets. Upon receipt of a packet, the

receiver checks for errors and returns an ACK/NACK accordingly. Given the transmission of a packet, say packet i , the transmitter must await an acknowledgment of the packet until after it finishes the transmission of up to $N-1$ subsequent packets (new or retransmitted), $1 \leq N \leq \infty$. If an ACK arrives at the transmitter during this waiting period, the corresponding packet is released from the transmitter's buffer and the N th subsequent transmission is a new packet. If a NACK or no acknowledgment is observed (time-out), the N th subsequent transmission is again packet i . The value N is referred to in the literature as the window size. In the receiver's buffer, every packet i is released if and only if all preceding packets have been released. Thus, at any given time, the receiver's buffer holds all packets that have been received correctly but for which at least one packet with a lower sequence number has not yet been received. The set of packets held by the receiver is termed the *buffer occupancy*.

The protocol described can be thought of as representing two types of physical systems.

1) *A Slotted Time Channel, Constant Packet Length, and Heavy Traffic*: The transmitter has an infinite supply of packets to transmit, all of which have the same length; a packet's transmission time equals one slot. The time-out length is constant and equals the round-trip propagation delay of $N-1$ slots. Hence, acknowledgments are expected to arrive exactly after $N-1$ slots; otherwise, a time-out is assumed. This implies that a packet is transmitted every N th slot until an ACK for it arrives at the transmitter. This model, commonly used in the study of satellite channels, has been adopted in most of the previous studies of ARQ protocols. With this interpretation, the model yields resequencing buffer occupancy and resequencing delay, measured in packets and slots, respectively.

2) *An Arbitrary Arrival Process and Variable Packet Length*: With this interpretation, packets are of arbitrary size and arrive at the transmitter according to an arbitrary process. The round-trip delay may be arbitrary, and the time-out, measured in number of transmissions, is of variable time length.

Also note that, with this interpretation, when a NACK for message i is received, the retransmission of the packet is delayed until $N-1$ other transmissions (counted from the last transmission of packet i) are completed. This contrasts with the possibility that the transmitter may retransmit an erroneous packet right after receiving its NACK. Therefore, with this interpretation we have a worst-case situation in which the buffer occupancy is independent of both the arrival process and packet length, nor does it depend on any other random variations in the network. In this respect, it is robust, a particularly important criterion for buffer design (since no statistics on the network may be available to the designer).

Note that with this second interpretation, there is no notion of real time, and the ARQ is based on a transmission count and window size. Thus only results considering buffer occupancy apply.

Consider the transmitted packets in blocks of N consecutive packets. The blocks are numbered and denoted by t , $t = 1, 2, \dots$. We shall refer to them as the window numbers.

Let $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$ be the identifiers of the packets that are transmitted during window t . Without loss of generality, we may assume that the packet identifiers are numbered in increasing order. That is, $X_1(t) < X_2(t) < \dots < X_N(t)$. The process $\{\mathbf{X}(t), t = 1, 2, \dots\}$ governs the evolution of the resequencing buffer occupancy.

To illustrate, suppose $N = 5$ and that packets number $1, \dots, 5$ are transmitted during window $t = 1$. Then $\mathbf{X}(1) = (1, 2, 3, 4, 5)$. Suppose also that the transmissions of packets 2, 4 fail, that is, the transmitter observes a NACK or a time-out for them. Under the SR ARQ protocol, packet 1 is released from the receiver's buffer, but packets 3 and 5 are held in the buffer and packets 2 and 4 are retransmitted in the next block. That is, $\mathbf{X}(2) = (2, 4, 6, 7, 8)$.

Assume that the probability of a packet failure (NACK or time-out) is p , $0 < p < 1$, and that the failures of packets are mutually independent. Let

$$D_i(t) = X_{i+1}(t) - X_i(t), \quad i = 1, 2, \dots, N-1;$$

$$D_N(t) = 1;$$

and

$$W_k(t) = \sum_{i=k}^N D_i(t), \quad k = 1, 2, \dots, N.$$

Furthermore, let

$$\mu_i(t) = E(D_i(t)), \quad \mu_{k,N}(t) = \sum_{i=k}^N \mu_i(t),$$

$$G_{k,t}(z) = E(z^{W_k(t)}), \quad |z| < 1.$$

When $t \rightarrow \infty$, the limiting random variables, expectations, and probability generating functions are denoted by D_i , W_k , μ_i , $\mu_{k,N}$, and $G_k(z)$, respectively. The existence of these limits is shown in the Appendix.

Let $X_{\min}(t)$ and $X_{\max}(t)$ be the minimum and maximum values, respectively, of the identifiers of these packets that have been transmitted up to and including window t but have not yet been released by the receiver.

From the protocol descriptions and the above assumptions, it immediately follows that, for every t ,

$$X_{\min}(t) = X_1(t), \quad X_{\max}(t) = X_N(t). \quad (1)$$

This equation simply states that, during every window t , the packets with the global maximum identifier and the global minimum identifier (not yet released by the receiver) are transmitted.

The number of buffers required at the beginning of window t is $W_1(t) - N$. For example, suppose that $N = 5$

and $(X_1(t), \dots, X_5(t)) = (7, 8, 11, 16, 17)$. From (1), this state indicates that all the packets whose identifiers are less than or equal to six have been released from the receiver's buffer. Furthermore, packets number 9, 10, 12, 13, 14, and 15 have arrived at the receiver and been positively acknowledged by it. Since packets are released according to the order of their identifiers, they have to be stored at the receiver at the beginning of window t . Hence the receiver's buffer occupancy is

$$W_1(t) - N = (18 - 7) - 5 = 6$$

packets. Let

$$B(t) = W_1(t) - N. \quad (2)$$

$B(t)$ is the buffer occupancy at the beginning of window t .

Remark 2.1: Note that the limiting distribution of the buffer occupancy at the beginnings of windows is the same as that observed at the beginning of an arbitrary packet transmission. This can easily be verified by observing that the evolution of the process $\mathbf{D}(t) = (D_1(t), D_2(t), \dots, D_N(t))$, $t = 1, 2, \dots$, is independent of the packet number to which we synchronize the window beginning. In the next section we evaluate the distribution of $W_1(t)$ under stationary conditions.

III. THE BUFFER OCCUPANCY AND THE RESEQUENCING DELAY

In this section we derive the probability-generating function (pgf) of the buffer occupancy under stationary conditions by using two different methods. The first method leads to a finite recursive computation of the pgf and all the moments. The second method leads to a representation of the pgf as an infinite summation from which the expected buffer occupancy can be derived in a simpler manner. We also derive the distribution of the resequencing delay.

A. The Probability-Generating Function: Recursive Formulas

The evolution of $(W_{N-i}(t), t \geq 1)$, $1 \leq i < N-1$, is governed by the following events.

- If there were $N-i+l$ NACK's during window t , $0 \leq l \leq i$, and if the $(N-i)$ th NACK was of packet $X_k(t)$, $N-i \leq k \leq N-l$, then $W_{N-i}(t+1) = W_k(t) + (i-l)$.
- If there were fewer than $N-i$ NACK's during window t , then $W_{N-i}(t+1) = i+1$.

Now, for $i = 0$, we clearly have

$$W_N(t+1) = W_N(t) = 1. \quad (3)$$

For $1 \leq i \leq N-1$, we have

$$W_{N-i}(t+1) = \begin{cases} i+1, & \text{with Pr } B(N, N-i-1), \\ W_k(t) + (i-l), & \text{with Pr } p \cdot b(k-1, N-i-1) \cdot b(N-k, l), \\ & 0 \leq l \leq i, N-i \leq k \leq N-l, \end{cases} \quad (4)$$

where $b(n, k) = \binom{n}{k} p^k (1-p)^{n-k}$ and $B(n, k) = \sum_{l=0}^k b(n, l)$.

In the Appendix we show that, for every N and $0 \leq p < 1$, the limiting distribution of $(W_1(t), W_2(t), \dots, W_N(t))$ exists. (It clearly exists for $p=1$.)

Letting $t \rightarrow \infty$, we have from (4)

$$G_{N-i}(z) = B(N, N-i-1)z^{i+1} + \sum_{l=0}^i \sum_{k=N-i}^{N-l} p \cdot b(k-1, N-i-1) \cdot b(N-k, l)z^{i-l}G_k(z), \quad 1 \leq i \leq N-1. \quad (5)$$

Let $B_n^*(z) = \sum_{i=0}^n b(n, i)z^{-i}$. From (3) and (5), we have

$$G_N(z) = z; \quad G_k(z) = \frac{z^{N-k} \left[B(N, k-1)z + p \sum_{i=k+1}^N b(i-1, k-1)B_{N-i}^*(z)G_i(z) \right]}{1 - p^k z^{N-k} B_{N-k}^*(z)}, \quad 1 \leq k \leq N-1. \quad (6)$$

From (6) we have a finite recursion to compute the probability-generating function $W_1, G_1(z)$. Note that all moments of W_1 can be derived from (6) in a recursive manner by taking the corresponding derivatives at $z=1$. This is done below for the second moment.

B. The First Two Moments — Recursive Formulas

Using a deductive reasoning we derive the first moment by a direct argument and the second moment by differentiating $G_k(z)$, $1 \leq k \leq N$, at $z=1$.

From (4) we have

$$\mu_{N-i, N} = (i+1)B(N, N-i-1) + \sum_{l=0}^i \sum_{k=N-i}^{N-l} p \cdot b(k-1, N-i-1) \cdot b(N-k, l)[\mu_{k, N} + (i-l)], \quad 1 \leq i \leq N-1.$$

Changing the summation order and using the identity

$$\sum_{l=i}^{N-1} p \binom{l}{i} p^i (1-p)^{l-i} + \sum_{l=0}^i \binom{N}{l} p^l (1-p)^{N-l} = 1, \quad (7)$$

we obtain

$$\mu_{N-i, N} = \left(i + (N-i)p^{N-i+1} - (1 - B(N, N-i-1)) \cdot N \cdot p + B(N, N-i-1) + \sum_{k=N-i+1}^N p \cdot b(k-1, N-i-1)[\mu_{k, N} + k \cdot p] \right) / (1 - p^{N-i}), \quad 1 \leq i \leq N-1. \quad (8)$$

For $i=0$ we have $\mu_{N, N} = 1$. Equation (8) provides a simple recursion for computing $E(W_1) = \mu_{1, N}$.

From the evolution of $D_i(t)$, $1 \leq i \leq N$, we also derive the following recursive formulas for μ_i , $1 \leq i \leq N$:

$$\mu_N = 1, \quad \mu_i = \frac{1}{1-p^i} \left(B(N, i) + \sum_{k=i+1}^{N-1} b(k, i)\mu_k \right), \quad 1 \leq i \leq N-1. \quad (9)$$

The probabilistic interpretation of (9) is used in Section IV to approximate the buffer occupancy distribution. Note also that (9) provides an alternative formula for $E(W_1)$.

The second moments $E(W_k^2)$ are obtained by differentiating $G_k(z)$ twice and using the relation

$$E(W_k^2) = \left. \frac{d^2 G_k(z)}{dz^2} \right|_{z=1} + \mu_{k, N}.$$

After simple algebraic manipulations, we have

$$E(W_N^2) = 1, \quad E(W_k^2) = \frac{[(N-k)p^k(1-p)]^2}{2(1-p^k)^2} + \frac{1}{1-p^k} \cdot \left((N-k)(N-k-1)p^k(1-p)^2 + (N-k)(N-k+1)B(N, k-1) + p \sum_{i=k+1}^N b(i-1, k-1) \cdot [E(W_i^2) - \mu_{i, N} + (i-k)^2 + (i-k) \cdot (2(N-i)(1-p)-1) + (N-i)(N-i-1) \cdot (1-p)^2 + 2\mu_{i, N}((N-i)(1-p) + (i-k))] \right) + \frac{2(N-k)(1-p)p^k}{(1-p^k)^2} \left((N-k+1)B(N, k-1) + p \sum_{i=k+1}^N b(i-1, k-1) \cdot [\mu_{i, N} + (1-p)(N-i) + (i-k)] \right), \quad 1 \leq k \leq N-1. \quad (10)$$

Again $E(W_1^2)$ can be computed recursively from (10).

C. The pgf and the First Moment: An Alternative Approach

Next we employ an alternative method to derive the pgf of the buffer occupancy. This method provides a different representation of the pgf, from which the expected buffer occupancy can be derived in closed form.

Here we consider the transmissions during each window in their original order, in contrast with the previous sections. In this representation every unsuccessful transmission in window t is repeated at the corresponding position in window $t+1$. Corresponding positions in consecutive windows will be referred to as columns.

Suppose that the Markov chain is under stationary conditions (the existence of which is proved in the Appendix) and consider the receiver's buffer at the end of an arbitrary transmission. Without loss of generality, we may assume that it is at the end of a window (see Remark 2.1). We say that at that instant, the *minimal valued packet* (mvp) which is transmitted at the current window is (n, k) if it is transmitted at column k and has undergone n unsuccessful transmissions. Note that the mvp is the packet that blocks its counterparts. The probability that the mvp is (n, k) is

$$p\{n, k\} = \begin{cases} qp^n(1-p^n)^{k-1}(1-p^{n+1})^{N-k}, & n > 0, \\ q^N, & n = 0, \end{cases} \quad (11)$$

where $q = 1 - p$.

This equation expresses the probability that the numbers of unsuccessful retransmissions of the packets from columns $1, 2, \dots, k-1$ and $k+1, \dots, N$ at the current window are strictly less than n and less than or equal to n , respectively. To see this, note that, if a packet from column j , $1 \leq j \leq k-1$, has n or more transmissions, its identifier is smaller than that of the mvp, which is a contradiction. A similar argument holds for a packet at column j , $k+1 < j \leq N$.

We now proceed to evaluate the distribution of the number of packets in the buffer, conditioned on the event that the mvp is (n, k) . What we want to find is how many packets of higher sequence number have entered the receiver's buffer since the mvp was first transmitted. This can be found by considering the individual contributions of each column.

Consider column j , $1 \leq j \leq k-1$, during the last n windows before the instant of observation. The condition that the mvp is (n, k) implies that at least one successful transmission has occurred at column j during these n windows. The first of these successful transmissions is a packet with a lower identifier than the mvp. This packet, therefore, is not at the receiver's buffer at that instant. Hence, the number of packets B_j in the buffer at the observation instant that have been transmitted on column j , equals the number of successful transmissions on that column during the last n windows minus one.

A similar argument holds for column j , $k+1 \leq j \leq N$, with the difference that at least one successful transmission

has occurred at the column during the last $n+1$ windows. Clearly, for $n=0$, $B_j=0$.

Since every transmission is successful with probability q , and since the successes are independent, the conditional distribution of B_j , given that the mvp is (n, k) , $n > 0$ is

$$\Pr\{B_j = m | n, k\} = \begin{cases} \frac{\binom{n}{m+1} q^{m+1} p^{n-m-1}}{1-p^n}, & \text{for } 1 \leq j \leq k-1, 0 \leq m \leq n-1, \\ \frac{\binom{n+1}{m+1} q^{m+1} p^{n-m}}{1-p^{n+1}}, & \text{for } k+1 \leq j \leq N, 0 \leq m \leq n. \end{cases}$$

Note that $\Pr\{B_k = m | n, k\} = 1$ for $m=0$, and 0 otherwise.

The pgf of B_j given (n, k) , is

$$G_{B_j}^{(n,k)}(z) = \begin{cases} 1, & n=0, 1 \leq j \leq N, \\ \frac{[(p+zq)^n - p^n]}{[z(1-p^n)]}, & n > 0, 1 \leq j \leq k-1, \\ \frac{[(p+zq)^{n+1} - p^{n+1}]}{[z(1-p^{n+1})]}, & n > 0, k+1 \leq j \leq N. \end{cases}$$

The conditional buffer occupancy under stationary conditions, B , is the sum of the independent conditional B_j 's. Therefore, the pgf of B , given (n, k) , $n > 0$, is

$$G_B^{(n,k)}(z) = \frac{1}{z^{N-1}} \left[\frac{(p+zq)^n - p^n}{1-p^n} \right]^{k-1} \cdot \left[\frac{(p+zq)^{n+1} - p^{n+1}}{1-p^{n+1}} \right]^{N-k}.$$

For $n=0$, $G_B^{(n,k)}(z) = 1$.

Averaging over the (n, k) values with the distribution from (11) and performing some standard algebraic manipulations, we derive the pgf of B :

$$G_B(z) = q^N + \frac{1}{z^{N-1}} \cdot \sum_{n=1}^{\infty} p^n \frac{[(p+zq)^{n+1} - p^{n+1}]^N - [(p+zq)^n - p^n]^N}{p^n + (z-1)(p+zq)^n}. \quad (12)$$

Note that the representation in (12) involves an infinite summation which is somewhat less convenient than the representation in Section III-A. However, by differentiating (12) at $z=1$, we obtain a simpler formula for $E(B)$:

$$E(B) = q \sum_{k=2}^N (-1)^k \binom{N}{k} \frac{p^{k-1}}{1-p^{k-1}}. \quad (13)$$

Remark 3.1: Clearly, the expected buffer occupancy is a function of p and N . As p increases, each packet is

expected to require more retransmissions and thus one would expect $E(B)$ to increase with p . This is indeed verified by computation. Therefore, to derive the worst expected buffer occupancy, we have to find $\lim_{p \rightarrow 1} E(B)$. Observe that, for $p=1$, $E(B)=0$ because none of the packets is received correctly.

Using L'Hopital's rule and standard combinatorial identities, we deduce from (13) that

$$\lim_{p \rightarrow 1} E(B) = N \sum_{k=2}^N \frac{1}{k}.$$

D. The Packet Resequencing Delay

Next we find the distribution of the resequencing delay under the first interpretation of our model (see Section II). Recall that this corresponds to the time elapsed from the end of the slot in which the packet was successfully transmitted to the end of the slot, where the last of the packets with lower identifiers are successfully transmitted. Note that, if a packet finds the buffer with no packets of lower identifiers, the waiting time is zero. This analysis assumes a heavy-traffic situation, hence providing upper bounds on the delay by virtue of an arbitrary arrival process.

Consider a tagged packet upon its successful transmission. Without loss of generality, we may assume that it is transmitted on column N . The probability that this packet has been transmitted m times until success is

$$P\{m\} = qp^{m-1}, \quad m \geq 1. \quad (14)$$

The packets that block the tagged packet in the buffer at its success time are those that were transmitted during the same window in which the tagged packet was first transmitted and have not been successfully received. The probability that the packet on column j , $1 \leq j \leq N-1$, blocks the tagged packet, given that the latter required m transmissions, is p^m . Thus, the conditional probability that at the instant of the success of the tagged packet, there are l packets with lower identifiers that have not been successfully received, is

$$P\{l|m\} = \binom{N-1}{l} p^{ml} (1-p^m)^{N-1-l}, \quad 0 \leq l \leq N-1. \quad (15)$$

These packets are referred to as the blocking packets. Given l blocking packets, the probability that there are k packets that require n more transmissions until success, while the rest of them require strictly fewer than n , is

$$P\{k, n|l\} = \binom{l}{k} (p^{n-1}q)^k (1-p^{n-1})^{l-k}, \quad 1 \leq k \leq l, n \geq 1. \quad (16)$$

Given (l, k, n) , the waiting time of the tagged packet, $H_{l,k,n}$, is

$$H_{l,k,n} = (n-1)N + U_k,$$

where U_k is the column where the k th last blocking packet is transmitted.

Clearly, the last k blocking packets are uniformly distributed over the set $\{1, 2, \dots, N-1\}$ and therefore

$$\Pr\{U_k = u\} = \frac{\binom{u-1}{k-1}}{\binom{N-1}{k}}, \quad k \leq u \leq N-1. \quad (17)$$

The waiting time of the tagged packet, H , assumes values $0, (n-1)N + u$, $n \geq 1, 1 \leq u \leq N-1$. From (14)–(17),

$$\begin{aligned} \Pr\{H = (n-1)N + u\} &= \sum_{m=1}^{\infty} P\{m\} \sum_{l=1}^{N-1} P\{l|m\} \\ &\quad \cdot \sum_{k=1}^l P\{k, n|l\} \Pr\{U_k = u\}; \\ \Pr\{H = 0\} &= \sum_{m=1}^{\infty} P\{m\} P\{l|m\}, \quad l = 0. \end{aligned}$$

A straightforward computation leads to

$$\begin{aligned} \Pr\{H = (n-1)N + u\} &= \frac{q}{p} \sum_{l=1}^{N-1} \binom{N-1}{l} \alpha(l) \beta(l, n, u) \\ \Pr\{H = 0\} &= \frac{q}{p} \alpha(0), \end{aligned}$$

where

$$\begin{aligned} \alpha(l) &= \sum_{i=0}^{N-1-l} \binom{N-1-l}{i} (-1)^{N-1-l-i} \frac{p^{N-i}}{1-p^{N-i}} \\ \beta(l, n, u) &= \sum_{k=1}^l P\{k, n|l\} \Pr\{U_k = u\}. \end{aligned}$$

The expected waiting time can be deduced simply from $E(B)$ by using Little's theorem. Note that the receiver's buffer can be viewed as a queue with an arrival rate of q packets per slot. Hence

$$qE(H) = E(B).$$

This completes the analysis of the resequencing delay distribution.

IV. BUFFER OCCUPANCY: APPROXIMATION FOR SMALL ERROR RATES

The probability-generating function of W_1 , which in turn yields the probability-generating function of the buffer occupancy, is handy for obtaining the moments. However, it is inconvenient for obtaining α -percentiles for the buffer occupancy (i.e., numbers k_α for which $\Pr(B > k_\alpha) = \alpha$). For this purpose we derive an approximation for the probability distribution of W_1 , which is particularly accurate when the probability of error is small.

First, we approximate the μ_i and then, the D_i 's and W_1 . Since for every k , $\mu_k \geq 1$, we have from (7) and (9)

$$\begin{aligned} (1-p^i)\mu_i &\geq \sum_{k=i+1}^{N-1} b(k, i) + B(N, i) \\ &= \frac{1-p^{i+1} - B(N, i)}{p} + B(N, i). \end{aligned}$$

Hence

$$\begin{aligned} \mu_i &\geq \left(B(N, i) + \frac{1 - B(N, i)}{p} - p^i \right) / (1 - p^i) \\ &\geq B(N, i) + \frac{1 - B(N, i)}{p}. \end{aligned} \quad (18)$$

Let $\tilde{\mu}_i = B(N, i) + [(1 - B(N, i))/p]$, $1 \leq i \leq N - 1$, and $\tilde{\mu}_N = 1$. The μ_i 's will be approximated by the $\tilde{\mu}_i$'s. The error of the approximation is given in Lemma 4.1.

The $\tilde{\mu}_i$'s have the following probabilistic interpretation, which will also motivate our approximation for the distribution of W_1 . For $1 \leq i \leq N$, define the following independent random variables:

$$\tilde{D}_i = \begin{cases} 1, & \text{with Pr } B(N, i) \\ Y, & \text{with Pr } 1 - B(N, i) \end{cases} \quad (19)$$

where Y is a geometric random variable whose probability of success is p . Clearly,

$$\tilde{\mu}_i = E(\tilde{D}_i), \quad 1 \leq i \leq N.$$

Lemma 4.1: For sufficiently small p 's, we have¹

$$\begin{aligned} \tilde{\mu}_i &\leq \mu_i \leq \tilde{\mu}_i + O(p), \quad 2 \leq i \leq N, \\ \tilde{\mu}_1 &\leq \mu_1 \leq \tilde{\mu}_1 + (e - 1) + O(p). \end{aligned}$$

Proof: The left side of the inequalities is derived in (18). From (9) and (18), we obtain

$$\mu_i - \tilde{\mu}_i \leq \sum_{k=i+1}^{N-1} b(k, i)(\mu_k - 1) + B(N, i) + O(p). \quad (20)$$

By induction on i , it is easy to verify that

$$\begin{aligned} 0 &\leq \mu_i - 1 \leq O(p^{2i-N}), \quad i \geq N/2 \\ 0 &\leq \mu_i - 1 \leq O\left(\frac{(1 - (1 - p)^{N-i})}{p \cdot i!}\right), \quad i < N/2. \end{aligned} \quad (21)$$

Combining (20) and (21) and using the Poisson approximation for $b(N, i)$, when p is small, yields the foregoing result.

Based on the probabilistic interpretation of \tilde{D}_i , $1 \leq i \leq N$, we approximate D_i by \tilde{D}_i and W_1 by $\tilde{W}_1 = \sum_{i=1}^N \tilde{D}_i$. Since the \tilde{D}_i are independent, one can easily compute α -percentiles for \tilde{W}_1 , which in turn result in α' -percentiles for W_1 , where α' is approximately equal to α .

To evaluate the approximation of W_1 we have to compare their probability generating functions. From (19) and the independence assumption we have

$$\tilde{G}_1(z) = E(z^{\tilde{W}_1}) = z^N \prod_{k=1}^N \left(B(N, k) + \frac{p(1 - B(N, k))}{1 - z(1 - p)} \right). \quad (22)$$

Now an evaluation has to be made regarding the differences $|\tilde{G}_1^{(k)}(z) - G_1^{(k)}(z)|$, $k = 0, 1, 2, \dots$, $1 - \epsilon < z < 1$,

¹ $c(p)$ is $O(p)$ if $\lim_{p \rightarrow 0} c(p)/p$ is a constant.

where $F^{(k)}(z)$ is the k th derivative of $F(z)$. Since this is analytically intractable, we shall compare the graphs of the generating functions. In Figs. 1-3 we draw the probability-generating functions for z close to 1 in three different scales. It can be observed that, for z close to 1, the derivatives of the two functions match quite well. Al-

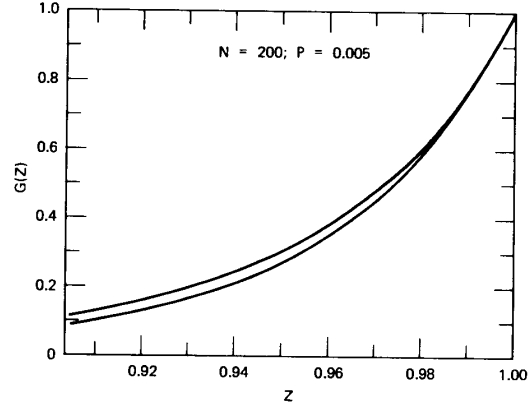


Fig. 1. $\tilde{G}_1(z)$ and $G_1(z)$ for $0.9 < z < 1$.

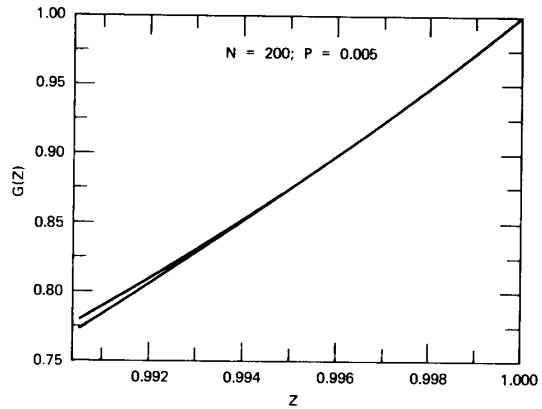


Fig. 2. $\tilde{G}_1(z)$ and $G_1(z)$ for $0.99 < z < 1$.

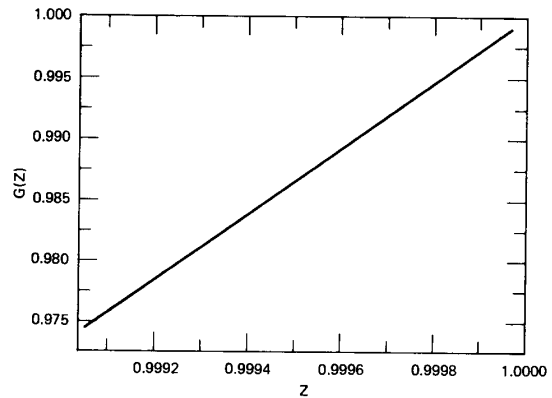


Fig. 3. $\tilde{G}_1(z)$ and $G_1(z)$ for $0.999 < z < 1$.

though the figures are for $N = 200$ and $p = 0.005$, we do get similar results for other N 's and small p 's.

V. NUMERICAL EXAMPLES

In Fig. 4 we present, for three different values of small p 's, the average buffer occupancy as a function of the window size N .

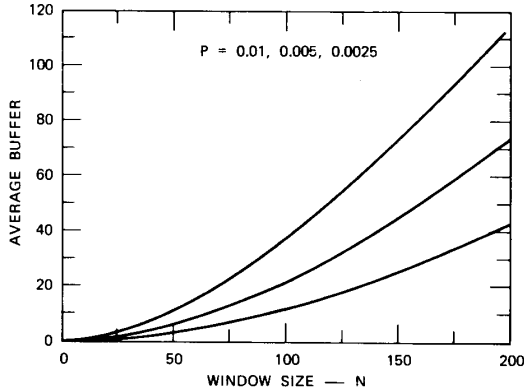


Fig. 4. Average buffer occupancy.

APPENDIX

THE EXISTENCE OF THE LIMITING DISTRIBUTION

Here we show that the Markov chain $W(t) = (W_1(t), W_2(t), \dots, W_N(t))$, $t \geq 1$ (as defined in Section III-A), is positive recurrent (ergodic) for every $0 \leq p < 1$ and N .

For every k , we first compute the average drift $g_k(w) = E[W_k(t+1) - W_k(t) | W_k(t) = w]$. From the definition of $W_k(t)$ and (4), we obtain the following:

$$\begin{aligned}
 g_k(w) &= (N - k + 1 - w)B(N, k - 1) \\
 &\quad + \sum_{l=0}^{N-k} \sum_{i=k}^{N-l} [W_i(t) + (N - k - l) - w] p \\
 &\quad \cdot b(i - 1, k - 1) \cdot b(N - i, l) \\
 &\leq (N - k + 1 - w)B(N, k - 1) \\
 &\quad + \sum_{l=0}^{N-k} \sum_{i=k}^{N-l} [W_k(t) + (N - k - l) - w] p \\
 &\quad \cdot b(i - 1, k - 1) \cdot b(N - i, l) \\
 &\leq (N - k + 1 - w)B(N, k - 1) \\
 &\quad + \sum_{i=k}^{N-k} \sum_{l=0}^{N-l} (N - k) \cdot p \cdot b(i - 1, k - 1) \\
 &\quad \cdot b(N - i, l) \\
 &= (N - k + 1 - w)B(N, k - 1) \\
 &\quad + (N - k)(1 - B(N, k - 1)) \\
 &= (N - k) + (1 - w)B(N, k - 1).
 \end{aligned}$$

Since $1 - w \leq 0$, we have

$$\sum_{k=1}^N g_k(w_k) \leq N(N-1)/2 + N \cdot b(N, 0) - b(N, 0) \sum_{k=1}^N w_k. \quad (23)$$

Let $\epsilon > 0$. From (23) it follows that, if $\sum_{k=1}^N w_k \geq N + [(\epsilon + N(N+1)/2)/b(N, 0)]$, then

$$\sum_{k=1}^N g_k(w_k) \leq -\epsilon. \quad (24)$$

Since for every c the set $\{(w_1, w_2, \dots, w_N) | \sum_{k=1}^N w_k \leq c\}$ is finite, it follows from Foster's criterion (see, e.g., [10]) and (24) that the Markov chain $(W(t), t \geq 1)$ is positive-recurrent. This implies the existence of (W_1, W_2, \dots, W_N) .

An intuitive explanation for the fact that the chain is positive recurrent is that, from every state of $(D_1(t), D_2(t), \dots, D_N(t))$, the chain returns to state $(1, 1, \dots, 1)$ in one step with probability at least $b(N, 0)$. This property of the chain also accounts for the finiteness of all the moments of W_1 . (The proof is standard.)

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