TDM Policies in Multistation Packet Radio Networks

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Abstract—**In this paper, we study a multistation packet radio network with m stations and a finite number of nodes n, that are using a conflict-free protocol to access the backbone network of stations through a shared channel. The goal is to derive an allocation of the channel time slots (TDM cycle), so that all transmissions will be conflict-free and some measure of performance (e.g., the expected total weighted throughput, the expected weighted holding cost) will be optimized.**

The methodology that is used is to bound the performance and to allocate the slots according to the golden ratio policy.

I. Introduction

A new *multistation network* model for hierarchical packet radio network has been recently introduced in [3], [4]. The model consists of a set of nodes that generate data packets and use a shared channel to transmit them to one of several stations which are connected by a backbone network.

The nodes represent the users which are geographically distributed and have a limited transmission range. The stations serve as concentrators and each of them concentrates the transmission activities of a different subset of nodes. The packets which are being originated by a given node might be destined to its corresponding station or to another node or station. In the latter, the stations act as repeaters.

There are three communication mediums in a multistation network.

1) The node-to-station medium (a common radio channel shared by all the nodes).

2) The station-to-station backbone network (another radio channel, wires, or fiber links).

3) The station-to-node medium (other radio channels with different bands to different stations).

A packet that is generated at a node and destined to another node is first transmitted to a station that concentrates the source node activities. This is done through medium 1) by employing some access algorithm. This station then, forwards the packet to a station that concentrates the destination node activities. This is done through medium 2). Finally, the packet is transmitted to the destination node through medium 3). When a packet is destined to a station, only medium 1) or mediums 1) and 2) are used.

In this study, we solely focus on the access algorithm in medium 1), and assume that the communication in the other two mediums is done independently and does not interfere with the access algorithm under investigation.

Various access protocols for medium 1) have been studied before. In [4], the regions of feasible node-to-station throughputs have been derived under the slotted ALOHA protocol. In [2], a two-station network under the CSMA and the BTMA has been studied. In [1], another class of multiaccess protocols—the collision resolution algorithms has been studied. In all the studies above, it has been assumed that the number of nodes in the network is infinite.

In this paper, we study a multistation packet-radio network with m stations and a finite number of nodes n, that are using a conflict-free access protocol to medium 1). We assume that the shared radio channel is slotted and all nodes are synchronized. Furthermore, all packet transmission times are exactly one slot and within the slot boundaries. Every node can be heard by at least one station; however, due to the broadcast nature of the transmissions, some of the nodes can be heard by more than one station.

Let H_j be the set of all nodes that can be heard by station j, $1 \le j \le m$. Since all nodes use the same radio channel, a packet transmitted by node $i \in H_j$ during an arbitrary time slot is successfully heard by station j, if and only if node i is the only node in set H_j that transmits during that time slot.

For every node i, let $T(i)$ be a preassigned set of stations responsible for concentrating the traffic from node i ($T(i)$ is obviously a subset of the stations that can hear node i). We define a transmission of node i to be *conflict-free* if all stations in $T(i)$ can hear the transmission successfully. Thus, our definition for conflict-free transmissions depends on how the set $T(i)$ is chosen. For instance, one may distinguish between *broadcast conflict-free* transmissions, in which every set $T(i)$ consists of all the stations that can hear node i; and *single conflict-free* transmissions, in which every $T(i)$ consists of a single station.

As an example, consider the network in Fig. 1. If broadcast conflict-free transmissions are required, then the simultaneous transmissions by nodes 5 and 11 are conflict-free—no other nodes can transmit at the same time without violating the broadcast conflict-free requirement [see Fig. 1(a)]. If single conflict-free transmissions are required and $T(5) = \{1\}$ and $T(9) = \{3\}$, then the simultaneous transmissions of nodes 5 and 9 are single conflict-free [see Fig. 1(b)].

The goal of this study is to derive an efficient allocation of the channel time slots among the nodes of a multistation network, so that all transmissions will be conflict-free in the sense defined above. In a network with a single station, the problem has been solved in [7] and [6]. In this study, we employ a similar methodology and show that the golden-ratio TDM policy provides an efficient allocation also in a multistation network. A similar problem, for a multihop network, has been addressed in [9] where spatial TDMA protocol has been introduced.

The rest of the paper is organized as follows. In Section II, we define general TDM policies and introduce two performance measures, the weighted expected throughput and the weighted expected holding cost, that are used to compare different TDM allocations. In Section III, we derive an upper bound for the expected weighted throughput and a lower bound for the expected weighted holding cost, under an arbitrary TDM policy. In Section IV, we introduce a TDM allocation that is generated by the golden-ratio policy [7], and compare its performance to the bounds in Section V.

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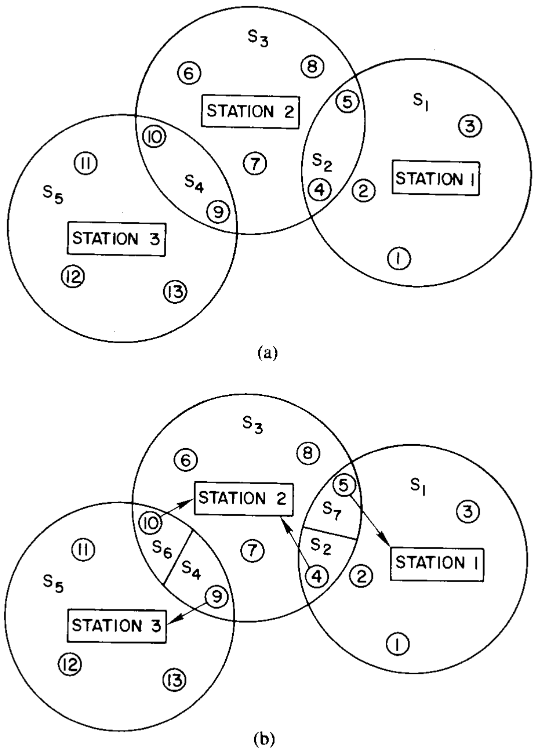

Fig. 1. A three-station network. (a) Broadcast conflict-free. (b) Single conflict-free.

II. A General Time Division Multiplexing (TDM) Policy

In this section, we introduce some notations from which we can easily define a general time division multiplexing (TDM) policy in a multistation network. Also, we define two performance measures (cost functions) associated with an arbitrary access protocol that we attempt to minimize.

A. Preliminaries

To define a general TDM policy we need the following notions. Let $\{1, 2, \cdots, n\}$ be the set of nodes and $\{S_j, 1 \leq j \leq s\}$ be a set of disjoint and inclusive subsets of $\{1, 2, \cdots, n\}$ that has the following property. "For each j, every two nodes in S_j can be heard by the same set of stations, as well as being concentrated by the same set of stations." A set S_j is referred to as a *node group*. For example, in Fig. 1(a) (broadcast conflict-free) the node groups are $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 5\}$, $S_3 = \{6, 7, 8\}$, $S_4 = \{9, 10\}$, and $S_5 = \{11, 12, 13\}$. In the network in Fig. 1(b) [single conflict-free], node 4 is concentrated by station 2, node 5 by station 1, node 9 by station 3, and node 10 by station 2. This yields the following node groups: $S_1 = \{1, 2, 3\}$, $S_2 = \{4\}$, $S_3 = \{6, 7, 8\}$, $S_4 = \{9\}$, $S_5 = \{11, 12, 13\}$, $S_6 = \{10\}$, and $S_7 = \{5\}$. It is clear that a simultaneous transmission of two or more nodes from the same node group is not conflict-free. Note that for broadcast conflict-free transmissions, $s \leq 2^m - 1$, and for single conflict-free transmissions, $s \leq m2^{m-1}$ where m is the number of stations in the network.

Furthermore, let $\{A^k, 1 \leq k \leq a\}$ be a set of subsets of $\{S_j, 1 \leq j \leq s\}$ with the following property. "For each k, if a single arbitrary node is selected from every S, $S \in A^k$, to transmit during the same time slot, then all transmissions are conflict-free." A set A^k is referred to as a *transmission group* and the set $\{A^k, 1 \leq k \leq a\}$ as a *transmission partition*. A transmission group A^k is *maximal*, if no other node outside A^k can transmit without interference during the same slot with any node in A^k. A transmission partition is *maximal* if every transmission group in the partition is maximal. In Fig. 1(a) [broadcast conflict-free], $A^1 = \{S_1, S_3, S_5\}$; $A^2 = \{S_1, S_4\}$; $A^3 = \{S_2, S_5\}$ is a maximal transmission partition. In Fig. 1(b) [single conflict-free] the maximal transmission partition is: $A^1 = \{S_1, S_3, S_5\}$; $A^2 = \{S_1, S_4\}$; $A^3 = \{S_1, S_6\}$; $A^4 = \{S_2, S_5\}$; $A^5 = \{S_4, S_7\}$; $A^6 = \{S_5, S_7\}$.

Finding a (maximal) transmission partition is the same as finding a (maximal) clique (completely connected subgraph, [5]) in an undirected graph, [9]. Indeed, let $G = (V, E)$ be the undirected graph whose vertices correspond to the node groups S_j. Furthermore, a pair of vertices v_i and v_j are connected by an edge if and only if the simultaneous transmissions of any node in S_i and any node in S_j is conflict-free. It is clear that a transmission group corresponds to a clique in this graph. Hence, finding a transmission partition is equivalent to finding the cliques, and finding a maximal transmission partition is equivalent to finding all maximal cliques. Construction of all maximal cliques in a graph is straightforward. In a graph with s vertices, there are up to 2^s maximal cliques. If the degree of every vertex in the graph is bounded by d, then the number of maximal cliques is bounded by $s \cdot d!$. In practice, the number of maximal cliques is significantly smaller than the bounds above (e.g., in the network of Fig. 1(a), we have only three maximal cliques).

To summarize, we note that the construction of a maximal transmission partition for a given multistation network is always feasible and the number of the transmission groups a is usually not too large.

The notions above will be used in Section II-B to define a general TDM policy.

Let $(V_i(t), t = 1, 2, \cdots)$, $1 \leq i \leq n$, be the arrival processes of packets at node i during successive time slots. We assume that for every node i, the process $(V_i(t), t = 1, 2, \cdots)$ forms a sequence of i.i.d. r.v.'s with finite expectation λ_i, and finite variance ξ_i. We will see below, that without loss of generality, we may assume that $\lambda_i < 1$, $1 \leq i \leq n$. Note that $\xi_i = 0$, $\lambda_i < 1$ and the fact that $(V_i(t), t = 1, 2, \cdots)$ are i.i.d., imply that $\lambda_i = 0$. Hence, we may assume that $\xi_i > 0$. We further assume that arrivals at distinct nodes are mutually independent. Each node has a given buffer capacity (finite or infinite) and all packets arriving during a slot, join the buffer at the slot end, if space is available.

B. The TDM Policies

Recall that all nodes of the network share the same access channel (e.g., frequency). However, they are distributed in a manner which allow several nodes (depending on their location) to transmit during the same slot without violating the conflict-free transmission requirement. The goal is to efficiently multiplex the channel time among the nodes, so that nodes will transmit their packets in a conflict-free manner (see Section I).

A general TDM policy can be defined by constructing a transmission partition $\{A^k, 1 \leq k \leq a\}$ and multiplexing the time slots among the transmission groups. Within the time slots belonging to each transmission group A^k, the slots are multiplexed among the nodes of each node group $S \in A^k$, independently of the multiplexing at other node groups. Note that we do not require a periodic TDM policy, nor that all nodes share the channel time equally.

For almost every practical cost structure (see below), it is suboptimal not to use a maximal transmission partition. Furthermore, whenever a slot is allocated to a transmission group A^k, it is suboptimal not to allocate the same slot to a single node in every node group $S \in A^k$. Suboptimal policies of this kind are excluded from our discussion.

Observe that every TDM policy can be defined with the single maximal transmission partition. Hereinafter, we consider only a maximal transmission partition.

C. The Performance Measures

A TDM policy π is determined by n disjoint sequences of slot numbers $(t_j^i, j = 1, 2, \cdots)$, $1 \le i \le n$, where t_j^i is the number of the jth slot allocated to node i.

Let $(d_j^i, j = 1, 2, \cdots)$, $1 \le i \le n$ be the interallocation times. That is, $d_1^i = t_1^i$ and $d_j^i = t_j^i - t_{j-1}^i$ for $j \ge 2$.

For definiteness, we assume that the system is empty at the beginning of slot number one. Since we will consider only policies under which the underlying Markov process is ergodic, the initial condition plays no role on the long-run average cost.

We consider two general types of performance measures (hereinafter—cost functions). The first one measures the long-run average weighted throughput of a system in which every node has a single buffer. The second one measures the long-run average weighted holding cost of a packet in a system where every node has an unlimited buffer capacity.

1) The Weighted Throughput: Here we assume that each node has a single buffer and a packet is lost if it arrives at a node whose buffer is full. Without loss of generality, we may assume that at most one packet can arrive at node i during a slot. Hence, λ_i is the probability that a packet arrives during one slot.

For every node i, let $S(i)$ be the node group that contains node i, and let $W(S(i))$ be the system gain of every packet that is transmitted by node i. The motivation for introducing different gains to different node groups is to have more general results and to allow flexibility in the way slots are allocated to different node groups (we elaborate on this in Section V). Using gains that are defined on a per node group basis (and not on a per node basis, for instance) is mainly done for mathematical tractability.

For every policy π and integer K, let $V_K^g(\pi)$ be the total expected gain of the system during the first K slots using policy π. Define the long-run weighted throughput of the system (under policy π)

$$\bar{V}^g(\pi) \overset{\text{def}}{=} \liminf_{K\to\infty} \frac{1}{K} V_K^g(\pi).$$

We are looking for a policy π that maximizes the long-run weighted throughput, $\bar{V}^g(\pi)$.

This problem has been studied in [7] for a single station network (a network with a single node group S). Using a Markov decision process formulation as in [7], it can be shown by the same proofs (mutatis mutandis) that in a multistation network the optimal policy (among all policies—not only TDM) is a periodic TDM policy with a finite period. That is, we may restrict our attention to TDM policies that are determined by n finite sequences $(d_j^i, j = 1, 2, \cdots, N_i)$, $i = 1, 2, \cdots, n$ where the period of the TDM cycle is N.

For such policies, $\bar{V}^g(\pi)$ is (see [7])

$$\bar{V}^g(\pi) = \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{N_i} W(S(i))[1-(1-\lambda_i)^{d_j^i}]. \tag{2.1}$$

To unify our mathematical treatment, we consider the cost function $\bar{V}^T(\pi)$ where

$$\bar{V}^T(\pi) = -\bar{V}^g(\pi) \tag{2.2}$$

and attempt to minimize instead of maximize.

2) The Weighted Holding Cost: Here we assume that each node has a buffer which is capable of storing an unlimited number of packets. Let $X_i(t)$ be the number of packets in the buffer of node i at the beginning of slot t. Let C_i be the cost per unit time of holding a packet in node i, and let $V_K^H(\pi)$ be the total expected cost until time K using policy π. Define the long-run average cost

$$\bar{V}^H(\pi) \overset{\text{def}}{=} \limsup_{K\to\infty} \frac{1}{K} V_K^H(\pi)$$

where by definition

$$V_K^H(\pi) \overset{\text{def}}{=} \sum_{t=1}^{K} E_\pi(C, X(t)) \tag{2.3}$$

with $(C, X(t)) = \sum_{i=1}^n C_i X_i(t)$. The expectation is taken with respect to the probability measure induced by policy π and the arrival process.

From (2.3), it follows that under stationary conditions

$$\bar{V}^H(\pi) = \sum_{i=1}^{n} C_i \bar{X}_i(\pi) \tag{2.4}$$

where $\bar{X}_i(\pi) = \lim_{T\to\infty} 1/T \sum_{t=1}^T E_\pi(X_i(t))$ is the stationary expected buffer occupancy at node i at an arbitrary slot beginning. With this cost function we are looking for a TDM policy that minimizes $\bar{V}^H(\pi)$. Note that for $C_i = 1/\lambda$, Little's Theorem implies that $\bar{V}^H(\pi)$ is the long-run average holding time of a packet.

It is easy to show, using the generalized Foster criterion [10], that the underlying Markov chain of the buffer occupancy at node i is ergodic if and only if $\lambda_i < \lim_{k\to\infty} k/\sum_{j=1}^k d_j^i \le 1$, whenever the limit exists. This explains our restriction $\lambda_i < 1$. We have to consider only ergodic policies, since under nonergodic policies the cost is clearly infinite. Thus, the cost function is indeed given in (2.4).

For a single station network, this problem has been studied in [6]. Note that under this cost function the optimality of the set of TDM policies is no longer valid as under the weighted throughput case. Hence, by considering only TDM policies we may exclude other good policies.

To summarize, we consider two types of multistation networks under TDM policies. In the first, every node has a single buffer and the cost function is given by (2.2). In the second, every node has an unlimited buffer capacity and the cost function is given by (2.4). The results of this paper remain valid for any cost structure which is a convex function of the costs in (2.2) and (2.4).

III. A Lower Bound on the Cost Function

Finding an optimal policy for $n > 2$ is known to be a difficult combinatorial problem, even for a single station network. Therefore, we adopt the methodology which has been used in [7] and [6]. Under each cost function, we derive tight lower bounds to the optimal cost. From this derivation, we also obtain a desirable allocation proportion $0 \le p_i^* \le 1$ for every node i. These proportions are optimal in an ideal system. That is, if we could allocate to each node i a proportion of p_i^* slots in equal interallocation distances $1/p_i^*$, and keeping the conflict-free transmission property, then the underlying TDM policy would attain the lower bound.

In this section, we derive the lower bounds and the desirable proportions for both cost functions. In Section IV, we show that the golden-ratio policy which implements these desirable proportions (see [7], [6]) provides costs which are very close to the lower bounds.

Observe that under any given TDM policy, the queues at distinct nodes do not interact and therefore they are independent. Hence, the cost functions in (2.2) and (2.4) are the same functions as in [7] and [6], respectively. Quoting the results from there (which are derived from the convexity of the cost function), we have for every TDM policy π whose allocation proportions are p_i, $1 \le i \le n$,

$$\bar{V}^T(\pi) \ge -\sum_{i=1}^{n} W(S(i))[p_i(1-(1-\lambda_i)^{1/p_i})], \tag{3.1}$$

$$\bar{V}^H(\pi) \ge \sum_{i=1}^{n} C_i \left[\frac{\xi_i}{2(p_i-\lambda_i)} - \frac{\lambda_i}{2}\right] \quad \text{for } \lambda_i < p_i. \tag{3.2}$$

The derivations of (3.1) and (3.2) are too lengthy to be repeated here. The interested reader is referred to the references above.

For $1/p_i$ integers, the expressions inside the brackets in (3.1) and (3.2) are the expected throughput and the expected buffer occupancy, respectively, under stationary conditions, using a TDM policy which allocates a slot to node i every $1/p_i$ slot. The lower bound holds also when $1/p_i$ are not integers.

A. A Convex Minimization Program

Equations (3.1) and (3.2) provide lower bounds for the cost of every feasible policy π whose allocation proportions are p_i, $1 \le i \le n$. Clearly, not every set of proportions is feasible. In a single station network ([7], [6]), the proportions have to satisfy the constraint $\Sigma_{i=1}^{n} p_i = 1$. In our multistation environment, the constraints on the proportions are more involved.

Let $\{A^1, A^2, \cdots, A^a\}$ be a maximal transmission partition and let $P(A^k)$ denote the proportion of slots allocated to transmission group A^k under an arbitrary TDM policy. From Section II, these proportions satisfy

$$\sum_{k=1}^{a} P(A^k)=1, \quad P(A^k) \ge 0, \quad 1 \le k \le a. \tag{3.3}$$

Furthermore, for every transmission group A^k and node group $S \in A^k$, let p_i^k, $i \in S$ be the proportion of slots that are allocated to node i from the slots that were allocated to A^k, under an arbitrary TDM policy. Again from Section II, for every $S \in A^k$ and $k = 1, 2, \cdots, a$, these proportions satisfy

$$\sum_{i \in S} p_i^k = 1, \quad p_i^k \ge 0, \quad 1 \le i \le n, \quad 1 \le k \le a. \tag{3.4}$$

The total proportion of slots that are allocated to node i, p_i, $1 \le i \le n$, and the proportions in (3.3) and (3.4), are related by

$$p_i = \sum_{k=1}^{a} P(A^k) p_i^k \tag{3.5}$$

where by convention $p_i^k \equiv 0$, if node i is not in any node group S, $S \in A^k$.

From (3.1)–(3.5), we can bound the cost function from below, by the solution of the following convex program. To avoid repetitions, denote by $\bar{V}(\pi)$ an arbitrary cost function and by $D_i(p_i)$, $1 \le i \le n$, arbitrary twice differentiable strictly convex functions. Furthermore, assume that

$$\bar{V}(\pi) \ge \sum_{i=1}^{n} D_i(p_i). \tag{3.6}$$

The weighted throughput and the weighted holding cost cases are obtained by taking $\bar{V}(\pi)$ as $\bar{V}^T(\pi)$ or as $\bar{V}^H(\pi)$, respectively, and $D_i(p_i)$, $1 \le i \le n$ as the summands in (3.1) or (3.2), respectively.

Consider now the following convex program:

$$\min_{P(A^k), p_i^k} \sum_{i=1}^{n} D_i(p_i) \tag{3.7a}$$

subject to

$$p_i = \sum_{k=1}^{a} P(A^k) p_i^k \tag{3.7b}$$

$$\sum_{k=1}^{a} P(A^k)=1, \quad P(A^k) \ge 0, \quad k=1, 2, \cdots, a. \tag{3.7c}$$

$$\sum_{i \in S} p_i^k = 1, \quad p_i^k \ge 0, \quad i \in S, \quad S \in A^k, \quad k=1, 2, \cdots, a. \tag{3.7d}$$

When the buffer capacity is unlimited, we have an additional constraint

$$p_i > \lambda_i, \quad 1 \le i \le n. \tag{3.8}$$

For the weighted throughput problem, we have to solve the convex program in (3.7), and for the weighted holding cost the convex program in (3.7) and (3.8). For the moment, consider only the convex program in (3.7). Observe that the objective function in (3.7a) is twice differentiable and strictly convex in $(p_i, 1 \le i \le n)$. In addition, we prove in the Appendix that the feasible region is a closed convex set. Therefore, there is a unique vector $(p_i, 1 \le i \le n)$ that solves the convex program, and we can use the Lagrangian principle and Kuhn–Tucker conditions to characterize the optimal solution (see e.g., [8]). Note also, that although the optimal p_i's are unique, it is not true in general that the optimal $P(A^k)$'s and p_i^k's are unique.

The convex program in (3.7) can be solved by standard algorithms such as the gradient projection method, active set methods or the primal method, [8]. Note, however, that the convex program can be quite large, mainly due to the variables $p_i^k \ge 0$, $i \in S$, $S \in A^k$, $k = 1, 2, \cdots, a$. The goals of the remainder of this section are to simplify the solution of the convex program in general, and to provide explicitly the optimal p_i^k's for our two special cases.

Let α, α_S^k, $S \in A^k$, $k = 1, 2, \cdots, a$, be the Lagrange multipliers corresponding to constraints (3.7c) and (3.7d), respectively. The Lagrangian function is

$$L(\boldsymbol{P}, \boldsymbol{p}, \alpha, \boldsymbol{\alpha}) = \sum_{i=1}^{n} D_i(p_i) + \alpha\left[1 - \sum_{k=1}^{a} P(A^k)\right] + \sum_{k=1}^{a} \sum_{S \in A^k} \alpha_S^k \left[1 - \sum_{i \in S} p_i^k\right] \tag{3.9}$$

where $\boldsymbol{P}$, $\boldsymbol{p}$, and $\boldsymbol{\alpha}$ are the vectors of the $P(A^k)$'s, p_i^k's, and the α_S^k's, respectively.

From Kuhn–Tucker conditions and the uniqueness of the optimal p_i's, every set of $P(A^k)$'s and p_i^k's that solve the following equations for some multipliers α and α_S^k, is optimal:

$$\sum_{S \in A^k} \sum_{i \in S} p_i^k D_i'(p_i) \begin{cases} = \alpha, & \text{if } P(A^k) > 0; \\ \ge \alpha, & \text{if } P(A^k) = 0. \end{cases} \qquad (k=1, 2, \cdots, a) \tag{3.9a}$$

$$P(A^k) D_i'(p_i) \begin{cases} = \alpha_S^k, & \text{if } p_i^k > 0; \\ \ge \alpha_S^k, & \text{if } p_i^k = 0. \end{cases} \qquad (i \in S, S \in A^k, k=1, 2, \cdots, a) \tag{3.9b}$$

$$\sum_{k=1}^{a} P(A^k) = 1. \tag{3.9c}$$

$$\sum_{i \in S} p_i^k = 1, \; S \in A^k, \; k=1, 2, \cdots, a. \tag{3.9d}$$

Here, $p_i = \Sigma_{k=1}^{a} P(A^k)p_i^k$ and $D_i'(\cdot)$ is the first derivative of $D_i(p_i)$ with respect to p_i.

Let $(P(A^k),\ k = 1, 2, \cdots, a)$, $\Sigma_{k=1}^{a} P(A^k) = 1$, be a feasible set of proportions allocated to the transmission groups. For every node group S, let $P(S) = \Sigma_{k:S\in A^k} P(A^k)$ (the total proportion of slots that are allocated to set S). In the Appendix, we prove the following Lemma.

Lemma 3.1: For every feasible set $(P(A^k),\ k = 1, 2, \cdots, a)$.

a) There exist optimal p_i^{*k}, $i \in S$, $S \in A^k$, $1 \le k \le a$, of the form $p_i^{*k} = q_i(S)$, that solve the convex program in (3.7a), (3.7b), and (3.7d).

b) For every S such that $P(S) = 0$, the optimal $q_i(S)$ can be chosen arbitrarily.

c) For every S such that $P(S) > 0$, the optimal $q_i(S)$ are the solution to

$$D_i'(P(S)q_i(S)) = \alpha_S, \quad i \in S \tag{3.10}$$

where α_S are some Lagrange multipliers.

From Lemma 3.1, we can partition the convex program problem in (3.7) into two simpler ones.

Problem I: For every node group S, find the optimal $(q_i(S),\ i \in S)$ for a given set of proportions $(P(A^k),\ 1 \le k \le a)$. That is, for every S find

$$\min_{q_i(S)} \sum_{i\in S} D_i(P(S)q_i(S)) \tag{3.11a}$$

subject to

$$\sum_{i\in S} q_i(S) = 1, \quad q_i(S) \ge 0, \quad i \in S. \tag{3.11b}$$

Note that we have s independent convex programs, one for each node group S. Furthermore, the solution to (3.11) is given by a solution to (3.10) and the constraints (3.11b). For every node group S, this problem is the same problem as in a single station network with a single node group S. For our two special cost functions, the weighted throughput and the weighted holding cost, we derive in Section III-B closed-form solutions to Problem I.

Problem II: Find the optimal $(P(A^k),\ 1 \le k \le a)$. That is, solve

$$\min_{P(A^k)} \sum_{i=1}^{n} D_i(P(S(i))q_i^*(S(i))) \tag{3.12a}$$

subject to

$$\sum_{k=1}^{a} P(A^k) = 1, \quad P(A^k) \ge 0, \quad k = 1, 2, \cdots, a \tag{3.12b}$$

where the $q_i^*(S)$'s are the solution to Problem I, and $S(i)$ is defined in Section II-C1).

By the Lagrangian principle, any solution to the following Kuhn–Tucker conditions is optimal.

$$\sum_{S\in A^k} \sum_{i\in S} q_i^*(S) D_i'(P(S)q_i^*(S)) \begin{cases} = \alpha, & \text{if } P(A^k) > 0; \\ \ge \alpha, & \text{if } P(A^k) = 0. \end{cases} \quad (k = 1, 2, \cdots, a) \tag{3.13a}$$

$$\sum_{k=1}^{a} P(A^k) = 1. \tag{3.13b}$$

The convex program in (3.12) can be solved by the standard methods mentioned earlier. Note that Problem II has a significantly smaller number of variables (only the variables $(P(A^k),\ 1 \le k \le a)$ appear here) than the original program (3.7) [that contains also the variables p_i^k, $i \in S$, $S \in A^k$, $k = 1, 2, \cdots, a$].

For the weighted throughput and weighted holding cost, the $q_i^*(S)$'s are explicitly given below.

B. The Maximal Weighted Throughput

When the optimality criterion is the expected weighted throughput, it follows from (3.1) and (3.11) that Problem I is:

For every node group S with $P(S) > 0$ find

$$\max_{q_i(S)} \sum_{i\in S} W(S)[P(S)q_i(S)((1-(1-\lambda_i)^{1/(P(S)q_i(S))})]$$

subject to

$$\sum_{i\in S} q_i(S) = 1, \quad q_i(S) \ge 0, \quad i \in S.$$

Kuhn–Tucker condition (3.10) becomes

$$W(S)(1-\lambda_i)^{1/P(S)q_i(S)}(1-\ln(1-\lambda_i)^{1/P(S)q_i(S)}) = \alpha_S, \quad i \in S.$$

This optimization problem is the same as in [7], for which the optimal $q_i^*(S)$'s are

$$q_i^*(S) = \frac{\ln(1-\lambda_i)}{\sum_{l\in S} \ln(1-\lambda_l)}. \tag{3.14}$$

Note that $q_i^*(S)$ is independent of $(P(A^1), \cdots, P(A^a))$ and of $W(S)$. Furthermore,

$$(1-\lambda_i)^{1/P(S)q_i^*(S)} = \bar{\lambda}(S)^{1/P(S)} \tag{3.15}$$

where $\bar{\lambda}(S) = \Pi_{i\in S}(1-\lambda_i)$.

From (3.1), (3.12) and (3.15), Problem II becomes

$$\max_{P(A^k)} \sum_{j=1}^{s} W(S_j)P(S_j)[1-\bar{\lambda}(S_j)^{1/P(S_j)}]$$

subject to

$$\sum_{k=1}^{a} P(A^k) = 1, \quad P(A^k) \ge 0, \quad k = 1, 2, \cdots, a.$$

Recall that $P(S_j) = \Sigma_{k:S_j\in A^k} P(A^k)$. Kuhn–Tucker condition (3.13) becomes

$$\sum_{S\in A^k} |S|\, W(S)[\bar{\lambda}(S)^{1/P(S)}(1-\ln \bar{\lambda}(S)^{1/P(S)})] \begin{cases} = \alpha, & \text{if } P(A^k) > 0; \\ \le \alpha, & \text{if } P(A^k) = 0. \end{cases} \quad (k = 1, 2, \cdots, a)$$

Corollary 3.1: For the maximal expected weighted throughput problem, we have to solve only Problem II. The optimal proportions p_i^k are given by $q_i^*(S)$ in (3.14), and they are independent of $(P(A^1), \cdots, P(A^a))$ and of $W(S)$.

C. The Minimal Weighted Holding Cost

When the optimality criterion is the expected weighted holding cost, it follows from (3.2), (3.8), and (3.11) that Problem I is:

For every node group S with $P(S) > 0$ find,

$$\min_{q_i(S)} \sum_{i\in S} \left[\frac{C_i \xi_i}{2(P(S)q_i(S)-\lambda_i)} \right],$$

subject to

$$\sum_{i\in S} q_i(S)=1, \quad q_i(s)\geq 0, \quad i\in S.$$

$$P(S)q_i(S)>\lambda_i, \quad i\in S.$$

Kuhn-Tucker condition (3.10) becomes,

$$\frac{C_i\xi_i}{2(P(S)q_i(S)-\lambda_i)^2}=\alpha_S, \quad i\in S.$$

This optimization problem is the same as in [6], for which the optimal $q_i^*(S)$'s are

$$q_i^*(S)=\left[\lambda_i+\frac{(C_i\xi_i)^{1/2}}{\sum_{l\in S}(C_l\xi_l)^{1/2}}(P(S)-\lambda(S))\right]\Big/P(S) \quad (3.16)$$

where $\lambda(S) = \Sigma_{i\in S}\lambda_i$.

Observe that only allocations with $P(S) > \lambda(S)$ are feasible, and that $q_i^*(S)$ is dependent on $(P(A^1), \cdots, P(A^a))$ through $P(S)$.

From (3.2), (3.12), and (3.16), Problem II becomes

$$\min_{P(A^k)} \sum_{j=1}^{s}\sum_{i\in S_k}\frac{(C_i\xi_i)^{1/2}\sum_{l\in S_j}(C_l\xi_l)^{1/2}}{2(P(S_j)-\lambda(S_j))},$$

subject to

$$\sum_{k=1}^{a} P(A^k)=1, \quad P(A^k)\geq 0, \quad k=1, 2, \cdots, a.$$

Kuhn-Tucker condition (3.13) becomes

$$\sum_{S\in A^k}\frac{\left(\sum_{l\in S}(C_l\xi_l)^{1/2}\right)^2}{2(P(S)-\lambda(S))^2}\begin{cases}=\alpha, & \text{if } P(A^k)>0;\\ \geq\alpha, & \text{if } P(A^k)=0.\end{cases} \quad (k=1, 2, \cdots, a)$$

Corollary 3.2: For the minimal expected weighted holding cost problem, we have to solve only Problem II.

IV. The Golden Ratio Policy

In Section III, we derived a lower bound for the cost function over all TDM policies, as well as desirable values for the $P(A^k)$'s and the p_i^k's. According to the explanation there, these values are optimal in an ideal system. In general, the lower bound cannot be attained since it is impossible to guarantee a uniform interallocation time for each node. However, by using the *golden ratio policy*, [7], we are able to allocate the slots in the best possible uniform manner. This fact has been well established in [7] and [6]. In this section, we first present the golden ratio policy and then show how it is applied to our problem. Finally, we give some numerical results which support our assertion regarding its efficiency.

A. Definition

Assume that we have N slots that are to be allocated among l entities in such a way that no two entities will be allocated the same slot. Let $x_i > 0$, $i = 1, 2, \cdots, l$, $\Sigma_{i=1}^{l} x_i = 1$, be the fraction of slots to be allocated to entity i. Let N and N_i, $i = 1, 2, \cdots, l$, be integers such that $\lfloor x_iN\rfloor \leq N_i \leq \lceil x_iN\rceil$ and $\Sigma_{i=1}^{l} N_i = N$ where $\lfloor x\rfloor(\lceil x\rceil)$ is the largest (smallest) integer smaller (larger) than or equal to x. (Note that $\lim_{N\to\infty} N_i/N = x_i$). Let $\phi^{-1} = (\sqrt{5}-1)/2 \sim 0.6180339887$ (ϕ^{-1} is the golden ratio), $\text{frac}(y) = y - \lfloor y\rfloor$, $a_j = \text{frac}(j\phi^{-1})$ and $A_N = \{a_j | j = 0, 1, \cdots, N-1\}$ (here, the tth smallest point in A_N is associated with the tth slot).

1) The Golden-Ratio Policy: The golden ratio policy is a TDM cycle of length N in which slots that correspond to the N_i points $\{a_j | \Sigma_{m=1}^{i-1} N_m \leq j < \Sigma_{m=1}^{i} N_m\} \subseteq A_N$ are allocated to entity i.

B. Application of the Golden Ratio Policy

The solution of the convex program in Section III yields the desirable fraction of slots to be allocated to each transmission group A^k, $k = 1, 2, \cdots, a$, and to every node $i \in S$, $S \in A^k$, within each transmission group. Recall that the latter proportions are denoted by p_i^{*k}, and are given by Lemma 3.1a) and (3.14) for the weighted throughput problem, and by Lemma 3.1a) and (3.16) for the weighted holding cost.

Once we have the optimal probabilities ($P^*(A^k)$, and p_i^{*k}, $i \in S$, $S \in A^k$, $k = 1, 2, \cdots, a$), we apply the golden ratio policy in two phases. In the first phase, the l entities are taken to be the partition groups A^k, $k = 1, 2, \cdots, a$, and the golden ratio is applied by taking x_i to be $P^*(A^k)$. As a result, we obtain a TDM cycle in which each slot is allocated to exactly one transmission group. In addition, the proportion of slots allocated to transmission group A^k, approaches $P^*(A^k)$ as the cycle length increases. Furthermore, the slots that are allocated to each transmission group A^k, are almost uniformly distributed over the TDM cycle [7].

In the second phase, we aggregate the slots allocated to transmission group A^k in the first phase, and for every $S \in A^k$ we take the nodes $i \in S$ to be the l entities, and apply the golden ratio by taking x_k to be $p_i^{*k} = q_i^*(S)$. As a result, each slot that has been allocated to transmission group A^k in the first phase, is now allocated to exactly one node $i \in S$, for every $S \in A^k$.

Clearly, the golden ratio policy provides a conflict-free TDM policy with all slots being allocated. Since the interallocation times d_j^i under the golden ratio policy are well defined, the weighted throughput is readily obtained from (2.1). As to the expected weighted holding cost, we need the phase method in order to evaluate the performance of the golden-ratio policy. This has been analyzed in [6], and all the necessary formulae are available there.

V. Numerical Examples

In this section, we provide some numerical examples for the performance of the golden ratio policy and compare it to the upper bound for the weighted throughput problem. We illustrate the results for two-station and four-station networks. In all the following examples, the length of the TDM cycle N is taken to be 1200.

A. A Two-Station Network

In the two-station network with broadcast conflict-free transmissions, we have three node groups: S_1 contains all nodes that are heard only by station 1 (say n_1 nodes), S_2 contains all nodes that are heard only by station 2 (say n_2 nodes), and S_3 contains all nodes that are heard by both stations (say n_3 nodes). The maximal transmission partition is $A^1 = \{S_3\}$ and $A^2 = \{S_1, S_2\}$. For this network, we consider five cases with different sets of parameters.

The first three are as follows: (i) $n_1 = 5$; $n_2 = 5$; $n_3 = 10$; (ii) $n_1 = 10$; $n_2 = 10$; $n_3 = 20$; (iii) $n_1 = 3$; $n_2 = 7$; $n_3 = 10$ where the arrival rates at all nodes are equal ($\lambda_i = \lambda\ \forall\ i$), as well as the weights of all sets S_i, $i = 1, 2, 3$. In case (iv), $n_1 = 5$; $n_2 = 5$; $n_3 = 10$; $\lambda_i = \lambda$ for $i \in S_1, S_2$, $\lambda_i = 4\lambda$ for $i \in S_3$ and all weights of the sets are equal. The last case, case (v) is similar to case 1 except for the weights where $W(S_1) = W(S_2) = 2W(S_3)$. The results are depicted in Figs. 2-4.

First, refer to Fig. 2 where $P(A^1)$ is depicted as a function

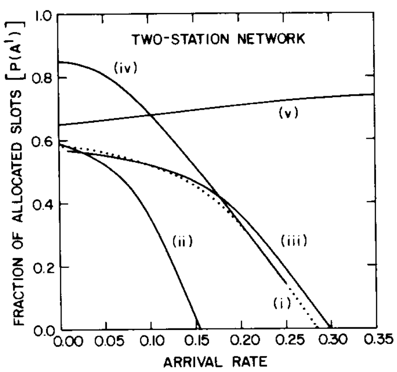

Fig. 2. Optimal proportions.

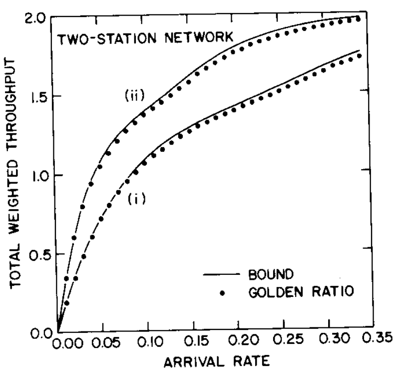

Fig. 3. Performance of the golden ratio policy.

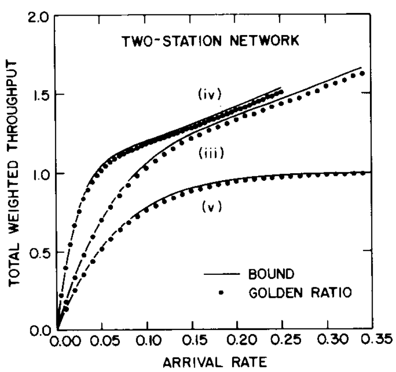

Fig. 4. Performance of the golden ratio policy.

of λ [note that here $P(A^2) = 1 - P(A^1)$]. The interesting phenomenon here is that for cases (i)–(iv), there exists some λ^* such that for $\lambda \geq \lambda^*$, $P(A^1) \equiv 0$. Note that this holds, even if the arrival rate at a node in A^1 is four times the arrival rate at a node in A^2 [case 4)]. The reason is that by allocating a slot to transmission group A^2, the throughput can be up as much as two units, while the throughput from an allocation to group A^1 is at most one unit. Note also, that λ^* becomes smaller as the number of nodes in S_1 and S_2 increases.

The above phenomenon indicates that the total throughput is not always the best measure of performance. It might result in an unfair allocation where certain nodes will never transmit. This is the primary reason for using the weighted throughput criteria in Section II-C1). By assigning different weights to different node groups [as we do in case (v)], we are able to

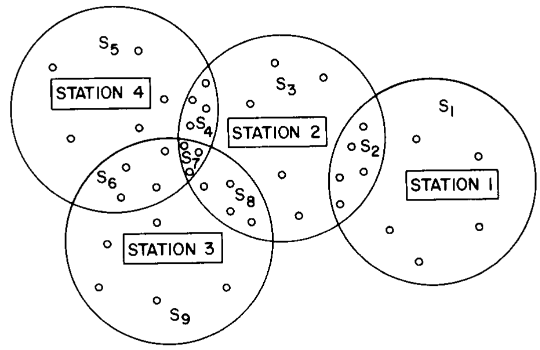

Fig. 5. A four-station network.

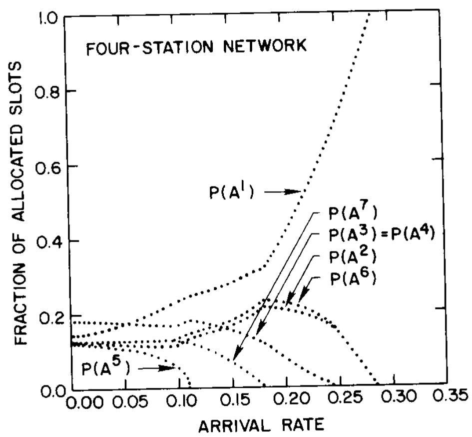

Fig. 6. Optimal proportions.

compensate small transmission groups by larger weights. As a result, we obtain in case (v), $P(A^1) > 0$ for any $0 < \lambda < 1$, since the maximum gain by any transmission is at most one unit (in Fig. 2 the results are depicted only up to $\lambda = 0.35$).

In Figs. 3 and 4, we depict the upper bound on the expected weighted throughput along with the expected weighted throughput obtained by using the TDM cycle that is generated by the golden ratio policy. We note that the performance of the golden ratio policy is extremely good; the difference between the bound (that cannot be attained in most cases) and the performance of the golden ratio policy is not greater than 2.5 percent.

B. A Four-Station Network

Here we consider the broadcast conflict-free network depicted in Fig. 5. Let n_i be the number of nodes in node group S_i (see Fig. 5). In this example, we have $n_1 = n_2 = n_3 = n_5 = n_9 = 5$; $n_4 = n_6 = n_8 = 4$; and $n_7 = 3$. The maximal transmission partition is $A^1 = \{S_1, S_3, S_5, S_9\}$; $A^2 = \{S_1, S_3, S_6\}$; $A^3 = \{S_1, S_4, S_9\}$; $A^4 = \{S_1, S_5, S_8\}$; $A^5 = \{S_1, S_7\}$; $A^6 = \{S_2, S_5, S_9\}$; $A^7 = \{S_2, S_6\}$. We assume that the weights of all node groups are equal and we consider two cases. In the first case, all arrival rates at the nodes are equal, and in the second case, there is some variability in the λ_i's.

First, consider the case where all arrival rates at the nodes are equal. The optimal probabilities $P(A^k)$ that yield the upper bound on the throughput are depicted in Fig. 6. Notice that as λ becomes larger, some probabilities are set to zero, and for $\lambda > 0.285$, only transmission group A^1 (that contains four node groups) is transmitting. The upper bound on the throughput, along with the performance of the golden ratio

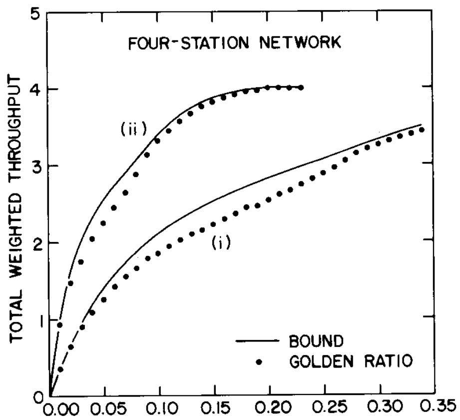

Fig. 7. Performance of the golden-ratio policy.

policy are depicted in Fig. 7 [case (i)]. We note that for low- and high-arrival rates the golden ratio has excellent performance. For medium values of arrival rates the golden ratio is about 10 percent worse than the bound. Case (ii) that is depicted in Fig. 7, corresponds to a set of arrival rates of the form $\lambda_i = \alpha_i \lambda$ for some set of α_i where the α_i's are taken so that the coefficient of variation of the arrival rates at different nodes would be significantly high. Again, we observe a very good performance of the golden ratio policy. Similar results are obtained for other set of parameters as well.

Appendix

A. Proof of Lemma 3.1

a) Let $(p_i^*, 1 \le i \le n)$ be the unique optimal solution to (3.7a), (3.7b), and (3.7d). For every S, if $P(S) > 0$ define $p_i^{*k} = p_i^*/P(S) = q_i(S), i \in S$; if $P(S) = 0$ define $p_i^{*k} = 1/|S| = q_i(S), i \in S$ where $|S|$ is the number of nodes in set S.

Since $P(S) = 0$ implies $p_i^* = 0, i \in S$, it follows the p_i^{*k}'s above are feasible and yield the optimal p_i^*'s.

b) This part is trivial.

c) As in the original convex program (3.7), every solution $(p_i^k, i \in S, S \in A^k, k = 1, 2, \cdots, a)$ to (3.9b) and (3.9d) with some multipliers α_S^k, are optimal. From part a), there is a strictly positive optimal solution $p_i^{*k} = q_i(S), i \in S$, that solve (3.9b) and (3.9d). Since $p_i^* = P(S)q_i(S)$, a solution $(q_i(S), i \in S)$ to

$$P(A^k)D_i'(P(S)q_i(S)) = \alpha_S^k, \quad i \in S, S \in A^k \quad \text{(A1)}$$

exists and is also optimal.

Now, clearly if $P(S) > 0$ then every solution to (3.10) is also a solution to (A1) and, therefore, optimal for the given proportions $(P(A^1), \cdots, P(A^a))$.

B. Proof that the Feasible Region is Convex

Let Ω be the feasible region as defined by (3.7b)–(3.7d) and (3.8), and let $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$ be a vector in Ω. To prove that Ω is convex, we have to show that for every $\boldsymbol{p}, \hat{\boldsymbol{p}} \in \Omega$ and $\alpha, 0 < \alpha < 1, \tilde{\boldsymbol{p}} \overset{\text{def}}{=} \alpha \boldsymbol{p} + (1-\alpha)\hat{\boldsymbol{p}} \in \Omega$.

For $i = 1, 2, \cdots, n$ let

$$p_i = \sum_{k=1}^{a} P(A^k)p_i^k, \quad \hat{p}_i = \sum_{k=1}^{a} \hat{P}(A^k)\hat{p}_i^k.$$

Define

$$\tilde{P}(A^k) = \alpha P(A^k) + (1-\alpha)\hat{P}(A^k), \quad k = 1, 2, \cdots, a.$$

For every $S \in A^k$ and $i \in S$ let $\tilde{p}_i^k = 1/|S|$ if $\tilde{P}(A^k) = 0$, and

$$\tilde{p}_i^k = \frac{\alpha p_i^k P(A^k) + (1-\alpha)\hat{p}_i^k \hat{P}(A^k)}{\tilde{P}(A^k)}, \quad \text{if } \tilde{P}(A^k) > 0.$$

Now, it is straightforward to verify that $\tilde{\boldsymbol{p}} = \Sigma_{k=1}^a \tilde{P}(A^k)\tilde{\boldsymbol{p}}^k$. Furthermore, $\tilde{P}(A^k), \tilde{\boldsymbol{p}}^k, k = 1, 2, \cdots, a$, satisfy conditions (3.7c) and (3.7d).

Also, it is apparent that if $\boldsymbol{p}$ and $\hat{\boldsymbol{p}}$ satisfy (3.8), so does $\tilde{\boldsymbol{p}}$. This completes the proof that Ω is convex.

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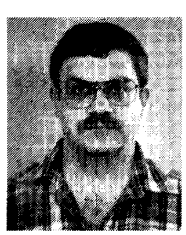

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